

2012

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This document is the publisher's version of an article published in:

Journal of Physics: Conference Series, vol. 360, (2012): 012030. doi: [10.1088/1742-6596/360/1/012030](https://doi.org/10.1088/1742-6596/360/1/012030)

Citation Information

Hinterleitner, F. and Major, Seth, "Plane gravitational waves and loop quantization" (2012). Hamilton Digital Commons.

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To cite this article: F Hinterleitner and S Major 2012 *J. Phys.: Conf. Ser.* **360** 012030

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Plane Gravitational Waves and Loop Quantization

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Abstract. Starting from the polarized Gowdy model in Ashtekar variables, the Killing equations characteristic for plane-fronted parallel gravitational waves are introduced in part as a set of first-class constraints, in addition to the standard ones of General Relativity. These constraints are expressed in terms of quantities that have an operator equivalent in Loop Quantum Gravity, making plane wave space-times accessible to loop quantization techniques.

1. Introduction

The motivation for studying plane gravitational waves is two fold: First, these waves are interesting objects on which to test possible quantum theories of gravity. Being homogenous in two directions and inhomogeneous in the third direction, the degree of difficulty of these models lies between the complicated theory of General Relativity (GR) and homogenous cosmological models. Second, the last decade has seen an abundance of conjectures on Lorentz invariance (LI) violation, dependence of the speed of light on energy, “Doubly Special Relativity”, and so on, see e.g. [1]. Some of these conjectures are inspired by the granularity of space predicted by Loop Quantum Gravity (LQG). So far there are plausibility arguments in favor of or against these conjectures, for example [2], but, to our knowledge, no calculation from an exactly solvable model. There is important prior work: plane gravitational waves were quantized [3]; LI has been investigated in the context of spherical symmetry [4], and [5] provides a derivation of gravitational wave dispersion in LQG in a cosmological context by perturbative methods. Gravitational plane waves appear to be simple enough for deducing whether LQG techniques yield dispersion, without further simplifications.

We start from a slightly more general system, the polarized Gowdy model in a form presented by Banerjee and Date [6, 7]. Like space with plane waves, this model is homogenous in two dimensions. The essential step in the reduction is to single out waves going into one direction and so to avoid colliding plane waves, which lead to complicated interaction processes and singularities. This reduction is carried out by means of a set of first-class constraints, so that the system becomes accessible to loop quantum techniques. See [8] for more details.

2. The polarized Gowdy model in Ashtekar variables

We assume homogeneity in the x, y plane and wave propagation in the z direction. We choose adapted spatial triads with one leg in the z direction and two arbitrary orthogonal vectors in the x, y plane. Densitized inverse triad variables are denoted by E^a_i , $a = x, y, z$ and $i = 1, 2, 3$. Following [6] we write $\mathcal{E} := E^z_3$ and introduce “polar” coordinates for the triad vectors in the x, y plane with the “radial” components E^x and E^y . The associated “angular” variable (the

same for E^x and E^y in the polarized Gowdy model) is pure gauge. They are fixed and disappear when the Gauß constraint is strongly imposed [6]. All variables depend only on z and the time variable t . The canonically conjugate diagonal elements of the Ashtekar connection (divided by the Barbero-Immirzi parameter γ) to E^x and E^y are the extrinsic curvature components K_x and K_y , the conjugate element to \mathcal{E} is denoted by \mathcal{A} . The fundamental Poisson brackets are

$$\{K_a(z), E^b(z')\} = \kappa \delta_a^b \delta(z - z'), \quad \{\mathcal{A}(z), \mathcal{E}(z')\} = \kappa \delta(z - z'). \quad (1)$$

$\kappa = 8\pi G_{\text{Newton}}$ is the gravitational constant.

In terms of these variables, and modulo the strongly imposed Gauß constraint, we have the following sets of first-class constraints of GR: The diffeomorphism constraint

$$C = \frac{1}{\kappa} [K'_x E^x + K'_y E^y - \mathcal{E}' \mathcal{A}], \quad (2)$$

(the prime denotes the derivative with respect to z) generating diffeomorphisms along the z -axis, and the Hamiltonian constraint

$$H = -\frac{1}{\kappa \sqrt{\mathcal{E} E^x E^y}} \left[E^x K_x E^y K_y + (E^x K_x + E^y K_y) \mathcal{E} \mathcal{A} - \frac{1}{4} \mathcal{E}'^2 - \mathcal{E} \mathcal{E}'' \right. \\ \left. - \frac{1}{4} \mathcal{E}^2 \left[\left(\ln \frac{E^y}{E^x} \right)' \right]^2 + \frac{1}{2} \mathcal{E} \mathcal{E}' (\ln E^x E^y)' \right]. \quad (3)$$

3. Reduction to plane waves

Plane waves are characterized by a null Killing vector field in the direction of propagation. With a lapse function $N(t, z)$ we have the space-time metric

$$ds^2 = -N^2 dt^2 + \mathcal{E} \frac{E^y}{E^x} dx^2 + \mathcal{E} \frac{E^x}{E^y} dy^2 + \frac{E^x E^y}{\mathcal{E}} dz^2. \quad (4)$$

For this metric a null vector field in the positive z -direction has the form $k^\mu = (\sqrt{E^x E^y / \mathcal{E}} k, 0, 0, \pm N k)$ with $k = k(t, z)$. With the minus sign chosen, the Killing equations $k_{\mu;\nu} + k_{\nu;\mu} = 0$ give rise to the following conditions

$$U_x = E^x K_x - \frac{1}{2} \mathcal{E}' - \frac{1}{2} \mathcal{E} \left(\frac{E^{y'}}{E^y} - \frac{E^{x'}}{E^x} \right) = 0, \quad U_y = E^y K_y - \frac{1}{2} \mathcal{E}' + \frac{1}{2} \mathcal{E} \left(\frac{E^{y'}}{E^y} - \frac{E^{x'}}{E^x} \right) = 0. \quad (5)$$

These two relations render the spatial diffeomorphism and Hamiltonian constraints equivalent,

$$C \approx -\sqrt{\frac{E^x E^y}{\mathcal{E}}} H = -\sqrt{g_{zz}} H.$$

4. Constraints

With test functions the expressions U_x and U_y become

$$U_a[f] := \int dz f(z) U_a(z). \quad (6)$$

The Poisson bracket structure may be expressed as

$$\begin{pmatrix} \{U_x[f], U_x[g]\} & \{U_x[f], U_y[g]\} \\ \{U_y[f], U_x[g]\} & \{U_y[f], U_y[g]\} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \int dz (f'g - fg') \mathcal{E}. \quad (7)$$

This matrix has a zero eigenvalue and is diagonalized by the combinations $U_+ = U_x + U_y$ and $U_- = U_x - U_y$, or explicitly

$$U_+ = E^x K_x + E^y K_y - \mathcal{E}', \quad U_- = E^x K_x - E^y K_y - \mathcal{E} \left(\ln \frac{E^y}{E^x} \right)'. \quad (8)$$

The algebra of these new constraints is

$$\{U_+[f], U_+[g]\} = \{U_+[f], U_-[g]\} = 0, \quad \{U_-[f], U_-[g]\} = 2 \int dz (f'g - fg') \mathcal{E}. \quad (9)$$

U_+ weakly Poisson-commutes also with C , and H :

$$\{U_+[f], C[g]\} = -\frac{1}{\kappa} U_+[f'g] \approx 0, \quad \{U_+[f], H[g]\} = \frac{1}{\kappa} U_+ \left[\sqrt{\frac{\mathcal{E}}{E^x E^y}} f'g \right] - H[fg] \approx 0. \quad (10)$$

Thus U_+ can be added as another first-class constraint. Together with the Poisson bracket relations of the standard constraints,

$$\begin{aligned} \{C[f], C[g]\} &= C[fg' - f'g], \quad \{C[f], H[g]\} = H[fg'], \\ \{H[f], H[g]\} &= C \left[(fg' - f'g) \frac{\mathcal{E}}{E^x E^y} \right], \end{aligned} \quad (11)$$

they establish the enlarged Poisson bracket algebra of the first-class constraints C , H , and U_+ . The constraint U_- can be written in terms of the three first-class constraints [8].

Concerning physical degrees of freedom counting, the gauge generators C and H reduce the number from originally three canonical pairs to one, when applied together with corresponding gauge conditions. The constraints $U_+(z)$ on the other hand, are not gauge generators, there are no gauge conditions associated with them, so they fix one function. Thus, when all constraints and gauge conditions are imposed, there is one free function left, like in the standard approach to plane waves. In this way the association of classical variables in canonical pairs gets lost. In the Dirac quantization approach however, one does not impose the constraints strongly, but as conditions on physical state functions, so in quantum theory the canonical structure remains untouched.

5. Preparation for quantization

In view of quantization it is important that U_+ can be given a meaning as a well-defined operator. Both $E^x K_x + E^y K_y$ and \mathcal{E}' are scalar densities that can be naturally integrated along z . The integral over some interval \mathcal{I} is

$$U_+[\mathcal{I}] = \int_{\mathcal{I}} dz (E^x K_x + E^y K_y) - \mathcal{E}_+ + \mathcal{E}_-, \quad (12)$$

where \mathcal{E}_{\pm} are the values of \mathcal{E} at the endpoints of \mathcal{I} . \mathcal{E} has a meaningful operator equivalent in the adapted LQG framework [7]. In analogy to full LQG the integral in (12) can be obtained as the Poisson bracket

$$\left\{ \int dx \frac{E^x K_x E^y K_y}{\sqrt{\mathcal{E} E^x E^y}}, \int dy \sqrt{\mathcal{E} E^x E^y} \right\} = \frac{1}{2} \int dz (E^x K_x + E^y K_y). \quad (13)$$

The first expression in the bracket is the restriction of the Euclidean Hamiltonian constraint to the x, y plane, the second one is the volume of a slice of space, constructed from a fiducial (unit) area in the x, y plane as basis and some perpendicular interval in the z -direction.

The analogs of both of them have operator equivalents in standard LQG, for the present case we find the corresponding operators in [7], equations (31) and (32). Now we are in a position to express all first-class constraints in terms of loop quantum operators, acting on one-dimensional spin network states, as demonstrated in [7].

6. Conclusion and outlook

In classical GR the existence of a null Killing vector field in the direction of propagation of gravitational plane waves guarantees dispersion-free propagation of such waves at a constant speed. In the present paper it was shown that a linear combination of two Killing equations, concerning area and volume expansion, can be expressed as a set of first class constraints in the polarized Gowdy model of [6]. The complementary linear combination U_- is dependent on the set of first-class constraints, so the latter one contains the full information about plane gravitational waves. An ongoing quantum investigation in the spirit of [7] should illuminate the issues of LI and dispersion of plane waves in LQG.

Acknowledgments

The work was supported by the Ministry of Education of the Czech Republic, contract no. MSM 0021622409.

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