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Seth Major
Hamilton College, smajor@hamilton.edu

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Embedded graph invariants in
Chern–Simons theory

Seth A. Major
Institut für Theoretische Physik, Universität Wien, Boltzmannasse 5, A-1090 Wien, Austria
Deep Springs College, Dyer NV 89010, USA

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Abstract

Chern–Simons gauge theory, since its inception as a topological quantum field theory, has proved to be a rich source of understanding for knot invariants. In this work the theory is used to explore the definition of the expectation value of a network of Wilson lines – an embedded graph invariant. Using a generalization of the variational method, lowest-order results for invariants for graphs of arbitrary valence and general vertex tangent space structure are derived. Gauge invariant operators are introduced. Higher order results are found. The method used here provides a Vassiliev-type definition of graph invariants which depend on both the embedding of the graph and the group structure of the gauge theory. It is found that one need not frame individual vertices. However, without a global projection of the graph there is an ambiguity in the relation of the decomposition of distinct vertices. It is suggested that framing may be seen as arising from this ambiguity – as a way of relating frames at distinct vertices. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Topological quantum field theory, developed by Atiyah [1] and Witten [2], is rooted in the desire to construct a framework independent of background structure. Chern–Simons gauge theory, being diffeomorphism invariant, provides an ideal test case. Witten [3] made the remarkable step of relating the vacuum expectation values of

1 E-mail: smajor@galileo.thp.univie.ac.at

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Wilson loops in Chern–Simons theory to the Jones polynomial [4]. This result was expanded to encompass more general groups and representations and observables based on projected graphs by Witten [5] and Martin [6]. Parallel to this work, techniques of non-perturbative, background independent quantization were developed [7-15]. Applied to gravity these techniques have led to a thorough understanding of quantum geometry [16,17]. It was discovered that in order to describe quantum geometry it is necessary to consider not only states based on projected Wilson lines with intersections but also to consider the full three-dimensional spatial structure of vertices; the tangent space structure of embedded graphs is required [16,17].

This paper, using the variational technique introduced in [18], generalizes the early results of Witten [5] and Martin [6] to embedded graphs with vertices of arbitrary valence and general tangent space structure. It is seen that the new invariants introduced here contain a dependence on the relative orientations of edge tangents. Further, it is seen in detail how the first-order variation exponentiates to the full, non-perturbative results expressed in terms of Temperley-Lieb recoupling theory [19] (in certain projections). With these techniques it is easily seen that the balanced networks of Barrett and Crane [20] are easily seen to arise from the first-order formalism. Finally, the variation calculations, accounting for the tangent space structure at vertices, strongly suggests that framing, originally introduced as a way to partially restore diffeomorphism invariance, is gauge.

This study comes from the confluence of the two approaches: the calculation of vacuum expectation values in Chern–Simons theory such is in Refs. [5,6,21-23] and the development of background independent quantization techniques [7-15]. The relation between this fields is actually tighter. More than a decade ago Kodama noticed that the Chern–Simons form is a formal solution to all the constraints of the vacuum theory with a cosmological constant [24]. As it is also natural to consider loops in this context as well, expectation values of Wilson loops in Chern–Simons theory have a dual role as “loop” transforms of this Kodama state in the connection representation of quantum gravity.

As we know that expectation values of loops in Chern–Simons theory require framing [3] the question arises as to what is the freedom associated with a framed vertex. The variational method provides a key to explore the definition of such singular graphs [26]. As shown in this paper the character of the invariant changes since an interplay of group and manifold structures determine its value. In fact, one recovers the Temperley–Lieb recoupling theory of Kauffman and Lins [19] tempered by a dependence on the embedding of the graph.

Motivation for this work also comes from a desire to have a more complete understanding of the loop representation for the Kodama state. Realizing that the loops had to be framed, a new representation of loop algebra of quantum gravity was introduced in Ref. [27]. In the framed loop representation, products of operators must also be defined. Two questions arise immediately: Is the product uniquely defined? If the product of operators is not unique, what is the freedom due to framing at a vertex? Thus it is natural to investigate the role of framing play at a vertex. From the perspective of the
framed loop representation or “q-quantum gravity,” it is interesting to find the vacuum expectation value of two loops (or generically graphs) which intersect. One of the goals of this work is to determine whether additional structure is required to specify this expectation value. This in turn should be reflected in the basic algebra of the operators in q-quantum gravity.

The main tools used in this study are based on spin networks. Originally introduced by Penrose as a method to solve the Four Color Theorem, and then as a combinatorial basis for space-time [25], spin networks have found a new life as a basis for 3-geometry states. In this new role spin networks are labeled graphs embedded in the three-dimensional spatial slice. The spin networks (or “spin nets”) resolve a long standing problem of the over completeness of the loop representation [8]. (A state space built simply from loops is subject to a number of identities called the Mandelstam identities.) This new basis solves these identities and comes with a bonus. The spin net basis is the eigenspace for geometric operators such as area [16] and volume [17]. The study of the kinematic states of quantum gravity using this basis I call, for the purpose of this paper, spin network geometry.

As Chern–Simons theory is a diffeomorphism invariant theory, it is not surprising that the vacuum expectation values of loops are functions of diffeomorphism invariant classes of knots and links (up to the subtleties of framing). However, the presence of singular knots or, in general, vertices significantly complicates the picture. One pleasing possibility is to define such graph invariants in terms of non-intersecting links. For instance, it may make sense to view intersecting knots as the limit points of non-intersecting knots; the freedom in taking the limit would encode precisely the freedom in the constructing intersections. One way to explore these ideas is through the study of Vassiliev invariants, in which one defines singular knots by associating to them an invariant defined by a difference of the possible limits of non-intersecting knots. For instance, the Vassiliev invariant associated a simple intersection is simply the difference between the over- and under-crossing decompositions, i.e. \( \langle \times \rangle = \langle \times \rangle - \langle \times \rangle \). For more than one vertex, this may be continued iteratively [21]. More generally, Vassiliev invariants allows one to associate to every knot an infinite sequence of rational numbers. This sequence divides up into finite sub-sequences which form vector spaces and, when indexed by an integer \( n \), are Vassiliev invariants of order \( n \). These invariants have a number of nice properties. They are related to knot polynomials: By replacing the variable of a polynomial invariant by \( e^x \), the coefficient of order \( n \) is a Vassiliev invariant of order \( n \). Vassiliev invariants of finite type vanish at order \( i \) for knots with more singular points, \( V_i(K^j) = 0 \) if \( j > i \) where \( j \) is the number of singular points. These form an algebra, so that two invariants of finite type, say \( i \) and \( j \), yield an invariant of the product \( ij \). Finally and spectacularly, it is conjectured that a complete set of knot invariants may be built from Vassiliev invariants [29]. The variational method used here is naturally associated to differences in non-intersecting knot invariants. It turns out that the lowest order results are Vassiliev invariants of finite type.

\( ^2 \) The structure for three-dimensional SU(\( N \)) Chern–Simons theories was also outlined in Refs. [5,6].
Recently Gambini, Griego, and Pullin, building on earlier work by Alvarez and Labastida [22], have proposed that Vassiliev invariants, generalized to include spin net states, are solutions to the Hamiltonian constraint of quantum gravity [30]. This construction is based on the idea that the framing dependence may be collected into an overall factor in the expectation value of Wilson loops. They find that the loop derivative operator is well defined when acting on Vassiliev invariants, suggesting that it may be possible perform dynamical calculations entirely in the spin net representation.

In Section 2 I give a review of the variational technique. This, as in the work of Labastida and Pérez [21], associates to every vertex a gauge invariant operator by analyzing the relations between families of (non-diffeomorphically related) graphs. Such relations are generated by taking the variation of the vacuum expectation value of loops. These loops depend on a parameter which interpolates between intersecting and non-intersecting loops. The variation allows one to construct an operator for vertices. Section 3 is devoted to applying this technique to graph invariants. As these operators a quite similar to the operators of spin net geometry, the same recoupling techniques may be used to find the expectation values. In Section 4 I present the results of these calculations. In some cases spin networks prove to be eigenspaces of the operators. In the final section I offer some concluding remarks.

2. Techniques of the variational calculation

In this section I give a self-contained review of the variational technique. Though the technique is widely known, the presentation serves to fix notation and to emphasize the embedding dependent elements of the calculation. In this way it becomes clear how one may generalize the technique to a projection-independent one. The method is an extension of those in Refs. [26,21], and [31]. It relies on several key properties of Chern–Simons gauge theory and the definition of spin nets which I mention before performing the variation. In the next section I collect the results and define gauge invariant operators for arbitrary graphs.

Let us consider Chern–Simons gauge theory on a smooth three-manifold \( \Sigma \) without boundary

\[
S[A] = \frac{k}{4\pi} \int_\Sigma \text{Tr} [A \wedge dA + \frac{2}{3} A \wedge A \wedge A]
\]  

(1)

in which \( A_a = A^i_a T^i \) is the Lie algebra-valued connection (\( a, b, c, \ldots \) are abstract spatial indices). The gauge group is, for this and the next section, taken to be a compact, semi-simple group \( G \). The trace is taken in the fundamental representation. The generators \( T^i, i = 1, 2, \ldots \dim G \), are normalized so that \( \text{Tr}[T^i T^j] = \frac{1}{2} \delta^i_j \). From the perspective of canonical quantum gravity, this action has another role as a state in the connection representation.

Kodama noted that there exists a state which formally satisfies all the constraints of vacuum canonical quantum gravity with cosmological constant [24]
\[ \Psi[A] = \exp(iS[A]), \]

where \( S[A] \) is the Chern–Simons action of Eq. (1) with \( k = 24\pi/\Lambda \) (\( \Lambda \) being the cosmological constant). In this perspective the manifold \( \Sigma \) is the spatial slice in the \( (3 + 1) \)-decomposition.

Given the gauge and diffeomorphism invariances of the theory, Wilson loops are natural observables. The vacuum expectation value of a (knotted) loop \( K \) is related to the Jones polynomial via [3]

\[ \langle W_K[A] \rangle = q^{-\frac{3}{2}w_K} J_K(q). \]

The Jones polynomial \( J_K(q) \) is the ambient isotopy polynomial invariant of \( q = \exp(Tri/k) \). (Omitting, for the present, the non-perturbative shift in \( k \). For more details see Appendix A.) The writhe, \( w_K \), is given in terms of the sum of crossings in a knot diagram of \( K \). In fact, the right-hand side is defined only for a projected knot in blackboard framing.\(^3\) The goal here is to generalize the observable in two ways. First, the variation technique reveals that it is possible to remove the projection dependence. Second, as is clear from both gauge theory and spin net geometry, it is wise to include observables based on graphs or spin nets.

Spin nets are embedded graphs labeled (or colored) with integers which represent the number of lines running along an edge and, equivalently, identify the irreducible representations carried by the holonomy. Every vertex contains a combination of Clebsch–Gordan coefficients which is called an intertwiner. For the purposes of this paper, a spin network \( \mathcal{N} \) consists of the triple \((G; i, n)\) of an oriented graph, intertwiners, and edge labels. The corresponding spin net state \( S \) is defined as

\[ \langle A \mid G; i, n \rangle = \prod_{e \in G} i_e \circ \bigotimes_{e \in V(G)} U(n_e)[A]. \]

These states are gauge invariant as the intertwiners are invariant tensors on the group. In more picturesque language, the state is gauge invariant because the intertwiners connect all the lines at each vertex.

To investigate the definition of the vertices, it will be convenient to analyze sub-spin nets. One may view the following analysis as cutting out a ball around a vertex, operating on it and then reinserting the result back into the graph. The operations are local so it is convenient to keep the intersections of the spin net and the vertex with the 2-sphere “fixed.” Interestingly, the variational method does not allow one to view these

\(^{3}\) Framing conventions are easy to understand with White’s theorem. This states that this self-linking number is the sum of the writhe and the twist [28]. Pictorially, self-linking is the winding number of the frame around the base loop; twist records the number of sides of the ribbon one sees in a projection (Möbius bands are ruled out); and writhe is given by the number of curls in a line. There are a number of framing conventions which fix writhe and/or twist including blackboard and standard framing. Blackboard framing, in which the frame is always normal to the knot in the plane of the blackboard, sets the twist to zero. By White’s theorem, the contribution to the self-linking number of a twist may be expressed as an equal number of curls. Standard framing requires the self-linking to vanish in any projection. It is naturally selected since it removes the explicit projection dependence. However, this choice only exists for certain manifolds including \( S^3 \) [3].
manipulations as solely operations on an abstract graph; one must also include tangent space information at the vertex. In this sense the calculation is tied to the manifold structure. The sub–spin nets I consider here consist of a vertex, incident edges, an intertwiner and the edge labels. This will be denoted as \( v = (v, e, i_v, n) \in (G; i, n) \).

To analyze subgraphs will be necessary to work with Wilson lines or holonomies. I will take a path \( \alpha \) to be an oriented, piecewise smooth map from the interval \( I = [0, 1] \) into \( \Sigma \). The composition of two paths will be denoted with \( \circ \) as in \( e_1 \circ e_2 \). I will take paths to be non self-intersecting.\(^4\) Associated to a path one has an holonomy

\[
U_\alpha[A] = P \exp \int_0^1 dt \dot{\alpha}^a A_a(\alpha(t)).
\]

Paths with boundary points identified are called loops. Wilson loops are simply the trace of holonomies based on loops. For a spin network \( \mathcal{N} \), the vacuum expectation value is defined by the functional integral

\[
\langle W_\mathcal{N}[A] \rangle = \int [DA] e^{iS[A]} W_\mathcal{N}[A].
\]

Two identities work hand in hand to make the variational calculation possible. The first shows that it costs curvature to differentiate the action with respect to the connection

\[
\frac{\delta}{\delta A_a(x)} S[A] = \frac{k}{8\pi} \epsilon^{abc} F_{bc}(x),\tag{2}
\]

where \( F_{ab} \) is the curvature of the connection \( A_a \). The second identity concerns the variation of paths. I take families of paths (frequently pairs of edges) of a spin network to be parameterized by a continuous parameter usually denoted \( u \). For some value of this parameter \( u \), the path intersects other edges.

Under the variation of a holonomy with respect to this parameter one discovers a magical property. The variation costs curvature. The change of an edge, \( e_u \) labeled by \( n \), of spin network state \( S \) is given by

\[
\frac{d}{du} U_{e_u} = \int_0^1 dt \dot{\alpha}^a(t) \dot{\epsilon}^b(t) F_{ab}^i(e(t)) \left[ U_{e(0,t)} T_{(n)}^i U_{e(t,1)} \right],\tag{3}
\]

where \( \dot{\alpha}^a \) denotes the derivative of the edge with respect to the parameter \( u \). The original spin net is recovered for one value of the deformation parameter \( u_0 \) when \( e_u|_{u=u_0} = 0 \). (I will not explicitly show the dependence of the edge \( e \) on \( u \) when it is clear from context.)

There are two distinct forms of variations of Eq. (3) [31]. In the first form, used for the decomposition of vertices, the paths depend on the parameter \( u \) which determines the “location” of the path relative to another, e.g. as in an “over-” and “under-” crossing. For this reason, I call \( u \) the “decomposition parameter.” Though the map \( e_u \) is continuous

\(^4\)There is no loss of generality; if, for instance, a loop had a single self-intersection then, expressed as a graph, the loop would have two edges.
Fig. 1. The four-valent vertex considered here has four edges $e_1, e_2, e_3,$ and $e_4$ labeled by integers $a$ and $b$. It is resolved into two crossing diagrams as shown in (b) for $u > 0$, $(U_{e_1 e_2 e_3 e_4})_+$, and (c) for $u < 0$, $(U_{e_1 e_2 e_3 e_4})_-$. The intersection (a) occurs when $u = 0$. The $u$-derivative of the edges, which determines the meaning of the over and under crossings is shown in (a).

in the manifold (like a homotopy transformation), given the topological nature of the theory, one expects that this variation of $\langle W \rangle$ to be discontinuous. In fact one finds that the variation of the expectation value is distributional. In the second form, in which the loops are parameterized such that the affect is that of a diffeomorphism, the expectation value is naively expected to vanish. As the theory requires framing, this is not always the case (as in Section 4.3). The difference of these two variations may be simply expressed as $u = u(t)$ for the first and $u = u(x)$ for the second.

The final piece of the variational calculation is the property of the holonomy

$$\frac{\delta}{\delta A_\alpha^a(x)} U_\alpha[A] = \int_0^1 dt \delta^\alpha(t) \delta^3(x, \alpha(t)) \left[ U_\alpha(0, t) T_i^a U_\alpha(t, 1) \right]$$

in which the path $\alpha$ carries the label $n$. The variation inserts a group generator in the $n/2$ representation into the holonomy at the point $x$.

As a first example of the variational calculation, consider the four-valent vertex, or double point, shown in Fig. 1. This is part of a larger spin net $\mathcal{N}$ so that the vertex $v = (v, e; i_1, n) \in \mathcal{N}$. Only the edges $e_1$ and $e_3$ are parameterized by $u$. Further, suppose that this dependence is coordinated so that when $u > 0$ ($u < 0$) the intersection is resolved into an over (under) crossing as seen when the vertex is projected along the $e^n$ direction. When $u = 0$ the edges intersect, forming the double point.

Using the identities of Eqs. (3) and (2) and integrating by parts, the variation of the expectation value $\langle U_v[A] \rangle$ may be expressed as

$$\frac{d}{dt} \langle U_v[A] \rangle = \frac{4\pi i}{k} \int [DA] e^{iS[A]}$$

$$\times \left[ \int_0^1 dt \delta^i_1(t) \delta^i_1(t) \epsilon_{abc} \frac{\delta}{\delta A_i^c(e_1(t))} (U_{e_1(0, t) T_i^a U_{e_1(t, 1)} U_{e_2} U_{e_3} U_{e_4})$$

$$+ \int_0^1 dt \delta^a_3(t) \delta^a_3(t) \epsilon_{abc} \frac{\delta}{\delta A_i^c(e_3(t))} (U_{e_1 U_{e_2} U_{e_3}(0, t) T_i^a U_{e_1(t, 1)} U_{e_4})$$

(5)
in which $U_e$ represents the whole holonomy along the edge $U_e(0,1)$. The variation produces a first-order (in $1/k$) result with a generator inserted into the network and differentiation with respect to the connection along the edges $e_1$ and $e_3$. In carrying out this variation, one clearly must make the critical assumption that the derivative commutes with the integration over the connection. It is also well to note that the integration over the loop parameter $t$ is only over the loop space where $\dot{e}_u(t)$ is non-vanishing.

The differentiation in Eq. (5) acts on all the holonomies of the incident edges. Thus, with the identity (4), Eq. (5) becomes $(n_1 = n_3 = a$ and $n_2 = n_4 = b$)

$$\frac{d}{du} \langle U_\phi \rangle = \frac{4\pi i}{k} \int [DA] e^{iS[A]} \sum_{j=1,3} \int_0^1 dt \dot{e}_j^a(t) \dot{e}_j^b(t) \epsilon_{abc} U_{e_j}(0,t) T^i_{(n_j)} U_{e_j}(t,1) \times \left[ \sum_{k=1,k \neq j}^4 \int_0^1 ds \dot{e}_k^c(s) \delta^3(e_j(t), e_k(s)) U_{e_k}(0,s) T^i_{(n_k)} U_{e_k}(s,1) \prod_{l \neq j,k}^4 U_{e_l} \right].$$

(6)

As complicated as this might appear, the structure is rather simple. One gauge generator is inserted on the edges which depend on the parameter $u$. The other generator is inserted in all other edge pairs, generating the second sum. Further, the delta-functions deflate these terms to terms specified by the condition $e_j(t) = e_k(s)$. For example, the terms

$$\int_0^1 dt \dot{e}_1^a(t) \dot{e}_1^b(t) \epsilon_{abc} \int_0^1 ds \dot{e}_3^c(s) \delta^3(e_1(t), e_3(s))$$

$$\times \left\langle U_{e_1}(0,t) T^i_{(n_1)} U_{e_1}(t,1) U_{e_3}(0,s) T^i_{(n_3)} U_{e_3}(s,1) \prod_{l \neq 1,3}^4 U_{e_l} \right\rangle$$

$$+ \int_0^1 dt \dot{e}_3^a(t) \dot{e}_3^b(t) \epsilon_{abc} \int_0^1 ds \dot{e}_1^c(s) \delta^3(e_3(t), e_1(s))$$

$$\times \left\langle U_{e_1}(0,t) T^i_{(n_1)} U_{e_1}(t,1) U_{e_3}(0,s) T^i_{(n_3)} U_{e_3}(s,1) \prod_{l \neq 1,3}^4 U_{e_l} \right\rangle$$

(7)

only differ by a sign when the condition is satisfied and thus cancel.

There are two classes of solutions to the above condition. For a single edge, $j = k$, there is a one-dimensional solution $s = t$. These singular terms require special care. Usually the line is split in two to give a loop and its frame. I will postpone discussion of this type of term until Section 4.3. The second class of solution, for $j \neq k$, lives only at a vertex. For instance, the first term of Eq. (6) may be expressed as

$$\frac{4\pi i}{k} \int_0^1 dt \dot{e}_1^a(t) \dot{e}_1^b(t) \epsilon_{abc} \int_0^1 ds \dot{e}_2^c(s) \delta^3(e_1(t), e_2(s))$$

$$\times \left\langle U_{e_1}(0,t) T^i_{(a)} U_{e_1}(t,1) U_{e_3}(0,s) T^i_{(b)} U_{e_3}(s,1) U_{e_3} U_{e_4} \right\rangle.$$
The $\delta$-function reminds us that the variation changes diffeomorphism equivalence class. The singular nature of this term is contained in the "volume" term

$$\int dt \int ds \epsilon_{abc} \dot{e}_1^a(t) \dot{e}_1^b(t) \dot{e}_2^c(s) \delta^3 (e_1(t), e_2(s)).$$

(9)

Nevertheless this result is useful. One may rewrite the delta-function as [21]

$$\delta^3 (e_1(t), e_2(s)) = \frac{1}{|\Delta_{(1,2)}(0,0,1)|} \delta(u) \delta(t) \delta(s - 1)$$

in which

$$\Delta_{(j,k)}(u, t, s) = \epsilon_{abc} \dot{e}^a_j(u) \dot{e}^b_j(t) \dot{e}^c_k(s)$$

for edges $e_j$ and $e_k$. Upon integrating over an interval around $u = 0$ the term of Eq. (8) may be simply expressed as

$$\int_{-\epsilon}^{+\epsilon} \frac{du}{u} \langle U_{e_i} \rangle = \frac{\pi i}{k} \kappa(1, 2) \langle U_{e_1} \left[T^i_{(a)} U_{e_1}\right] \left[U_{e_2} T^i_{(b)}\right] U_{e_1}\rangle$$

(10)

in which

$$\kappa(j, k) = \frac{\Delta_{(j,k)}(u_0, 0, 1)}{|\Delta_{(j,k)}(u_0, 0, 1)|}.$$

(11)

The sign factor $\kappa(j, k)$ is only defined at the intersection when $u = u_0$. I also use the convention that $\int_0^1 dx \delta(x) = \frac{1}{2}$. It is clear from the structure of the operator of Eq. (10) that the affect of the variation is simply to act with a left or right invariant vector field on the incident edges (indicated with square brackets above). The handedness is determined by the orientation of the edge. It is clear that both group structure and tangent space structure determine the variation of the invariant. Thus, for this type of solution, the result may be expressed as insertion of generators at the vertex times a sign. In the next section I present the form of the variation for this vertex and more general vertices. In each case they correspond to lowest order Vassiliev invariants.

3. Gauge invariant operators for arbitrary vertices

In Ref. [21] gauge invariant operators were constructed for knots with planar double points. From the perspective of spin net geometry, in which states of the theory based on planar vertices have vanishing volume expectation values [17], it is clear that one would like to generalize the construction to non-planar higher valence vertices. This section contains an analysis for these cases. The next section contains an evaluation of some of these operators on spin network states using SU(2) group structure.

As an example of the gauge invariant operators for vertices, consider the four vertex calculation presented in the last section. The terms on the right-hand side of Eq. (6), depending on the nature of the volume term, fall into two classes. Here I will only
study the terms between distinct edges, postponing the “self-interaction terms” until Section 4.3. As noted in the last section the two terms with the edge pair $e_1$ and $e_3$ cancel, due to the opposite signs. Thus, Eq. (6) reduces to

$$\int_{-\epsilon}^{\epsilon} \frac{d}{du} \langle U_\psi \rangle = \langle U_{e_1o_3e_2o_4} \rangle_+ - \langle U_{e_1o_3e_2o_4} \rangle_-$$

$$= \frac{\pi i}{k} \left( \kappa(1,2) + \kappa(1,4) + \kappa(3,2) + \kappa(3,4) \right)$$

$$\times \left\langle \left[ U_{e_3} T^{i}_{(a)} U_{e_1} \right] \left[ U_{e_2} T^{i}_{(b)} U_{e_4} \right] \right\rangle. \quad (12)$$

It is understood that the holonomies are over complete edges and the brackets indicate the composition of the holonomies. The sign is determined by the decomposition parameter and the tangents of the incident edges. For a simple vertex such as this four-valent one, all the terms of Eq. (6) combine into one. For higher order vertices this is not the case. Clearly the tangent space structure and the choice of the vector field $\vec{e}_u(t)$ determine the decomposition; the sign terms collect as overall factors on each term. In this four-valent vertex, the four terms may take nine possible values which are realized by different configurations of the incident edges. For instance, the vertex shown in Fig. 1 has an overall factor of 4. Similarly in the planar case where the edge pairs have the same tangents, all the signs are identical and this simply reduces to one term [21]

$$\pm 4 \frac{\pi i}{k} \left\langle \left[ U_{e_3} T^{i}_{(a)} U_{e_1} \right] \left[ U_{e_2} T^{i}_{(b)} U_{e_4} \right] \right\rangle. \quad (13)$$

The overall sign is determined by the direction of the vector field associated with the parameter $u$. Since the sign also changes on the left-hand side of Eq. (12), there is no additional freedom in determining the planar 4-vertex.

As this last point will become more important later, it is worth expanding on. The vector field $\vec{e}_u$ determines the deformation of the vertex. One may lift one edge pair using such a vector field as is done here. One may also “slide” a vertex along an edge by taking a vector at the vertex $\vec{e}_u(0)$ which is parallel with one of the tangents at the vertex. In the case of a simple 4-vertex such a slide is redundant since the lines are reparameterization invariant; the spin net state does not change. However, it illustrates the edge of the region of equivalent decomposition vectors. For the 4-vertex of Fig. 1, all vectors $\vec{e}_u|_{u=0}$ in the tangent space volume designated by the triple $\vec{e}_1, \vec{e}_4, \vec{e}_3^{-1}$ yield equivalent decompositions in that they are diffeomorphic [32]. The compliment also yields the same decomposition.

For graphs with more than one vertex, this procedure of lifting edges iteratively may be applied to all the vertices in the graph [21]. The procedure may also be applied to higher valent vertices. In this case, for a 2n-valent vertex one uses $n - 1$ variations of this type. The lowest order in $1/k$ of this decomposition is then also of order $(n - 1)$

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5 For higher valence vertices, even planar ones, such a slide changes the valence of the original vertex and creates more vertices. In these cases, the variation is non-vanishing.
Fig. 2. The six-valent vertex $v = (v, e; i_1, n)$ is shown with an intertwiner tree, (b). The over-crossings in (b) only indicate the nature of the connections in the intertwiner. The dotted line indicates that the diagram inside has no spatial extent. The paths $e_1 \circ e_4$ and $e_2 \circ e_5$ are parameterized by $u$ and $v$, respectively, and varied in the calculation. The first term of the variation is given in (c).

on account of the integration by parts for each variation. It is worth noting that gauge invariant operators of this type may only be associated to vertices with even valence. Otherwise one would be left with an open edge and the resulting state would no longer be gauge invariant.

Before giving the general case, I will present the derivation for a six-valent intersection. Consider the vertex shown in Fig. 2 with six edges and an internal tree with labels $i_1, i_2$ and $i_3$. This vertex may be decomposed in at least three ways. For instance, the vertex may be decomposed first into a four-valent vertex created from $e_2, e_3, e_5$, and $e_6$ by lifting the edge pair $e_1 \circ e_4$. The decomposition may be completed by lifting the edge pair $e_5 \circ e_2$. As with the Vassiliev invariants for double points, the intersection may be expressed as a signed sum of over- and under-crossings. Performing such a decomposition by first lifting the line $e_1 \circ e_4$ one finds that the eight terms begin with

\[
\frac{d}{du} \langle U_v \rangle = \frac{4\pi i}{k} \int ds \int dt \epsilon_{abc} \hat{e}^a_i(t) \hat{e}^b_j(t) \delta^3(e_1(t), e_2(s)) \times \langle i_v \circ U_{e_1}(0, t) T_i^{(a)} U_{e_1}(t, 1) U_{e_2}(0, s) T_i^{(b)} U_{e_2}(s, 1) U_{e_6} \rangle + \ldots
\]

in which the holonomies and generators are composed with the intertwiner. To finish the decomposition one can lift a second line, here chosen to be $e_2 \circ e_5$. This line is parameterized by $v$ and the vector field $\hat{e}_v(t)$ is taken to be in the same direction as the vector field associated with $u$. After integration in the two parameters $u$ and $v$ the result becomes

\[
\left\langle \begin{array}{c} \times \\ \times \\ \times \\ \times \end{array} \right\rangle - \left\langle \begin{array}{c} \times \\ \times \\ \times \end{array} \right\rangle - \left\langle \begin{array}{c} \times \\ \times \end{array} \right\rangle + \left\langle \begin{array}{c} \times \end{array} \right\rangle = \left(\frac{4\pi i}{k}\right)^2 \left[ \kappa^2(\{1, 4\}, \{2, 5\}) \langle i_v \circ U_{e_1} T_i^{(a)} U_{e_2} U_{e_3} U_{e_5} U_{e_6} \rangle + \kappa(\{1, 4\}, \{3, 6\}) \langle i_v \circ U_{e_1} T_i^{(a)} U_{e_2} U_{e_3} U_{e_6} \rangle + \kappa(\{1, 4\}, \{2, 5\}) \langle i_v \circ U_{e_1} U_{e_2} T_i^{(a)} U_{e_3} U_{e_5} U_{e_6} \rangle + \kappa(\{1, 4\}, \{2, 5\}) \langle i_v \circ U_{e_1} U_{e_2} T_i^{(a)} U_{e_3} T_i^{(c)} U_{e_5} U_{e_6} \rangle \right]
\]
\[ +\kappa(\{1,4\},\{3,6\})\kappa(\{2,5\},\{3,6\}) \left( i_v \circ U_e T_{(a)}^i U_{e_1} U_{e_2} T_{(b)}^j U_{e_3} U_{e_4} T_{(c)}^{(i)} U_{e_5} \right) \]

(15)
in which the signs have been collected with
\[ \kappa(\{i,j\},\{k,l\}) = \kappa(i,k) + \kappa(i,l) + \kappa(j,k) + \kappa(j,l) \]
and \( T^{(ij)} := T^i T^j + T^j T^i \) symmetrization defined without a numerical constant.

A few remarks are in order. First, the diagrams on the left-hand side are a schematic representation of the decomposition. The plane of the projection is given by the vector \( \epsilon_u(i_1) \) (\( i_1 \) is the value of the loop parameter at the vertex). Different choices of the vector fields \( \epsilon_u(t) \) and \( \epsilon_v(t) \) yield different decompositions. For a planar vertex, there are three independent ways to decompose the vertex corresponding to permutations of the three paths from which the vertex is built. The choice of decomposition is made by which edges are lifted and limits of the decomposition parameters. (The order of the lifting does not influence the result as the variations commute.) All possible decompositions may be so generated. Here, the limits of the \( u \) integration are chosen to be less than the \( u \) integration, i.e. \( \epsilon_u > \epsilon_v \). Second, this result is explicitly second order as is expected for a Vassiliev invariant of finite type. Third, the tangent space structure and the group structure separate in each term. Fourth, no framing was required in the decomposition because the variation did not produce volume terms which had to be regulated. Of course, part of this was by fiat since the edge self-linking terms were neglected (these will be discussed in Section 4.3). Nonetheless, this calculation does indicate that a vertex decomposition does not need such a regularization. Fifth, as both the operators and the intertwiners are invariant tensors on the group, the operators are gauge invariant.

The iterative procedure used on the six-valent vertex may be carried out on higher valence vertices as well. An even valence vertex may be completely decomposed in this manner while odd valence vertices have a minimally trivalent decomposition. The general decomposition of a 2n-valence vertex is conveniently expressed if an incident edge \( i \) is numbered so that its partner (the edge which joins to the first when the vertex is decomposed) is numbered \( n+i \). It will also be convenient to label the pairs by the first edge. For instance, the edge \( e_1 \) has partner \( e_{(n+1)} \) and the pair \( e_1 \circ e_{(n+1)} \) is labeled by the index 1. The general form of the operator is clear from the two previous calculations. Generators are inserted in all the lifted edge pairs. The other \( (n-1) \) generators may be inserted in any of the remaining \( (n-1) \) edge pairs. Since these permutations are all distinct one may index them with one parameter \( m \). There are \( N = (n-1)^{n-1} \) possibilities. For the 2n-vertex \( v_{2n} = (u, e; i_v, n) \) one has
\[ D^{(n-1)} \langle U_{v_{2n}} \rangle = \left( \frac{\pi i}{k} \right)^{(n-1)} (n-1) \sum_{m=1}^{N} \kappa_m \prod_{e=1}^{n} \left( i_v \circ U_e \phi_{(n)}^{ij...k}(m) U_{(e+n)} \right). \]

(16)
The map \( \phi_{(n)}(m) \) gives the generator insertions
\[ \phi_{(n)}^{ij...k}(m) = T_{(n)}^{(i)} T_{(n)}^{(j)} \ldots T_{(n)}^{(k)}. \]
Each possible term, indexed by $m$, contains a symmetrized set of generators. The sign factor is given by the product

$$\kappa_m = \prod_{l=1}^{n-1} \kappa \left\{ \{l, (l+n)\}, \{\rho_m(l), (\rho_m(l) + n)\} \right\}$$

in which $\rho_m(l)$ labels the edge pair induced by the permutation $m$. For example, the last term of Eq. (15) indexed by $m = 4$ would have $\rho_4(1) = \rho_4(2) = 3$ while all other values of $\rho_4$ vanish. The overall structure of Eq. (16) is easy to see: All lifted pairs have generators. The sum is over all the possible insertions of the remaining generators while the product is over the incident edges. A decomposition of an entire graph would include a second product over all vertices.

The vacuum expectation value of the operators of Eq. (16) gives an invariant for the singular graph with a $2n$-valent vertex. This invariant may be expressed as a signed sum of non-intersecting knots or links. Since the result is minimally of $(n-1)$ order, the invariant vanishes for all graphs with an additional crossing (either in the vertex under consideration or at another site on the graph). Thus, this provides an instance (in a slightly different setting) of the theorem by Birman and Lin which states that the $n$th order coefficient of the expansion of a polynomial knot invariant is a Vassiliev invariant of order $n$ [29].

For singular knots and links these operators may be given a graphical form. In these diagrams, each component of the link is represented by a circle with marked points. These points represent the values of the loop parameter(s) where an intersection occurs. The generators are inserted at these points. The contractions of the group index are represented as a dashed line. For the knot shown in Fig. 3, one operator,

$$\langle \text{Tr} \left[ U_{s_1} T^{i_1 j_1} U_{s_2} T^k U_{i_1} T^{i_1} U_{j_1} T^{j_1} U_{k} \right] \rangle$$

is shown in part (b). The pair of indices $ij$ at $s_1$ are symmetrized. This is represented in the figure as a pair of crossed dashed lines. For multicomponent links the distinct circles will, generically, have dashed lines between them. Since each term in the decomposition is a distinct operator, each diagram defines a distinct configuration. Different links may be classified according to their diagrams. The diagrams may be seen to label a product of $N_v$-dimensional vector spaces (where $N_v$ is as above for a $2n$-valence vertex). A link may be identified by its configuration. These diagrams are also useful for computing invariants of expectation values of spin net states with a flat connection, such as the numerical invariants of Ref. [22].

It is intriguing to note that, between vertices of the same graph, there exits no natural relation between the vector fields associated to the decomposition parameters at distinct vertices. Of course, in a global projection of the entire graph, there is a natural choice of the vectors fields, such out of or into the page. However, in the absence of such a projection, the relation must be consistent with an arbitrary projection at each vertex. What one might seek is a way to leave the projection choice arbitrary at each vertex and yet still have a consistent system. Such a system is reminiscent of a gauge.
Fig. 3. The operator associated to the singular knot given in (a) may be represented with a diagram, e.g. (b). The operator represented is $\langle \text{Tr}[U_{e_1 T^i} U_{e_2 T^j} U_{e_3 T^k} U_{e_4 T^l} U_{e_5 T^m} U_{e_6 T^n} U_{e_7 T^p}] \rangle$ – one of four terms. The tangent space dependence has been omitted.

4. Evaluating $SU(2)$ vertex operators

For $SU(2)$, it is relatively simple to express the invariant operators of the last section in terms of spin networks. The techniques are similar to those used in the geometric operators of quantum gravity (see Refs. [16,17]). As in the case of these operators it is convenient to use the methods of recoupling theory. In this section I shall present the calculations using the diagrammatic methods of Kauffman and Lins [19]. Of course, the recoupling theory is for the group $SU(2)$ not for the quantum group $SU(2)_q$. When the spin network basis is an eigenbasis for the gauge invariant operators (as they are in some cases), the variation operator may be exponentiated to give the full series. The advantage is immediate. The simple relation between Temperley–Lieb recoupling theory and the “classical” or “binor” conventions of regular $SU(2)$ recoupling theory, in which ordinary recoupling theory is obtained when $A \to -1$ [19], suggests that the variation yields the lowest order result of the invariant, “The classical result gives the quantum exponent.”

To find the action of the $SU(2)$ graph observables it is necessary to introduce a spin net decomposition of the gauge generators. This extension of the simple relation

$$[T^i]^B_A [T^i]^D_C = -\frac{1}{2} \left( \delta^D_A \delta^B_C - \frac{1}{2} \delta^B_A \delta^D_C \right) = \frac{1}{2} \left( \sum_{\sigma, \xi, \eta} \left( \frac{a_{a(b)}^{\sigma \xi \eta}}{\delta^A_\alpha \delta^B_\beta} \right) \right)$$

(for $SU(2)$ using binor conventions) for single lines becomes

$$[T^i_{(a)}]^B_A [T^i_{(b)}]^D_C = \frac{1}{4} \sum_{c=|a-b|}^{a+b} \left( -1 \right)^{a(a,b)} a_c(a,b) \frac{\Delta_c}{\theta(a,b,c)}$$

in which

$$a_c(a,b) = \frac{1}{2} \left[ a(a+2) + b(b+2) - c(c+2) \right].$$

The recoupling quantities $\theta(a,b,c)$ and $\Delta_c$ are given by Eqs. (A.8) and (A.5) in the Appendix. Dotted ovals in these diagrams represent the recoupling at the vertex. The

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5 The norm used in the identity differs from the normalized spin networks of Ref. [14] by a factor of $\sqrt{A_c}$. 
identity of Eq. (18) may be applied to the gauge invariant operators of the last section. I will give results for the 4-valent, 3-valent, and 2-vertices before turning to a brief discussion of the general case.

4.1. Four-valent vertices

With the above identity in hand, one may compute the action of the operators of the last section on the spin net states. I will only evaluate the operator for planar vertices here. The extension of these results to arbitrary four-valent vertices is straightforward.

The variation of the planar four-valent vertex of Eq. (13) with the intertwiner \( \chi \) indexed by \( i \) is

\[
\left\langle \begin{array}{c}
\chi_a \\
\chi_b
\end{array} \right\rangle - \left\langle \begin{array}{c}
\chi_a \\
\chi_b
\end{array} \right\rangle = \frac{\pi i}{2k} \sum_c (-1)^{a_c(a,b)} \times a_c(a,b) \frac{\Delta_c}{\theta(a,b,c)} \left[ \left\langle \begin{array}{c}
\chi_a \\
\chi_b
\end{array} \right\rangle + \left\langle \begin{array}{c}
\chi_a \\
\chi_b
\end{array} \right\rangle \right]
\]

\[
= \frac{\pi i}{k} (-1)^{a(a,b)} \times \left( \frac{a(a+2) + b(b+2) - i(i+2)}{2} \right) \left\langle \begin{array}{c}
\chi_a \\
\chi_b
\end{array} \right\rangle
\]

in which the identity (A.18) was used in the first line. The recoupling calculation of the first term is simple with the use of Eq. (A.9) while the recoupling for the second terms is more involved, using identities (A.10), (A.16), and (A.17). It is interesting to note that spin nets are an eigenbasis for the variation. This suggest that one ought to exponentiate the first-order result. The exponentiation is not unique. However, by examining the sum over intertwiners, one can match these results with those of Temperley–Lieb recoupling theory.

When evaluating the operators such as the four-valent vertex operator of Eq. (13), the vertices are labeled by intertwiners. If we are to determine the expectation value of the product of spin net states, which have a pointwise intersection, the calculation is not defined from the group theory standpoint until some sort of intertwiner is specified. When an intertwiner is not specified it is most natural to sum over the possible intertwiners as in Eq. (A.19). The variation then yields

\[
\left\langle \begin{array}{c}
\chi_a \\
\chi_b
\end{array} \right\rangle - \left\langle \begin{array}{c}
\chi_a \\
\chi_b
\end{array} \right\rangle = \frac{\pi i}{2k} \sum_i (-1)^{(a+b-i)/2} \left[ a(a+2) + b(b+2) - i(i+2) \right] \times \frac{\Delta_i}{\theta(a,b,i)} \left\langle \begin{array}{c}
\chi_a \\
\chi_b
\end{array} \right\rangle
\]
(which is identical to the variation without specifying an intertwiner). This agrees with the result from Temperley–Lieb recoupling theory in so much as the first-order coefficients are equivalent: Using Eq. (A.19) the two crossings may be expressed in terms of the intertwiner

\[
\left\langle \begin{array}{c}
\bar{a} \\
\bar{b}
\end{array} \right\rangle - \left\langle \begin{array}{c}
\bar{a} \\
\bar{b}
\end{array} \right\rangle = \sum_{i} (\lambda_i^{ab} - (\lambda_i^{ab})^{-1}) \frac{\Delta_i}{\theta(a, b, i)} \left\langle \begin{array}{c}
\tilde{a} \\
\tilde{b}
\end{array} \right\rangle.
\]

Expanding the \( \lambda \) coefficients to first order one finds

\[
\left\langle \begin{array}{c}
\bar{a} \\
\bar{b}
\end{array} \right\rangle - \left\langle \begin{array}{c}
\bar{a} \\
\bar{b}
\end{array} \right\rangle = \sum_{i} (-1)^{(a+b-i)/2} \left( \frac{\pi i}{k} a_i(a, b) \right) \frac{\Delta_i}{\theta(a, b, i)} \left\langle \begin{array}{c}
\tilde{a} \\
\tilde{b}
\end{array} \right\rangle + O(1/k^2),
\]

which matches the variational result of Eq. (20). In this comparison one learns that the first-order variation captures the first-order term and the overall sign. Thus, to match the conventions of Temperley–Lieb recoupling theory, one must exponentiate only the first-order dependence on the labels, not the overall sign.

When evaluating the vacuum expectation value of a product of operators based on knots which intersect, there is not a unique choice of intertwiners for the new vertices. The most natural solution is to sum over possible intertwiners. The above calculation shows that a product of operators based on intersecting loops will be a sum of operators. Thus, in the perspective of \( q \)-quantum gravity, a product of intersecting loop operators will generically give a sum over admissible states. It may be best to use a new basis or a new set of operators based on the set of eigenstates of the \( T_q \) operator of Ref. [27].

It is interesting to note that the variational technique produces only a restricted set of invariants associated with signed sums of vertex decompositions. For instance, the relation for single lines \(^7\)

\[
\begin{array}{c}
\times
\end{array} = -\frac{1}{A^2 + A^{-2}} \left( \begin{array}{c}
\times
\end{array} + \begin{array}{c}
\times
\end{array} \right)
\]

used in Ref. [27] as the “deformed Mandelstam identity” simply does not appear. This “averaging decomposition” clearly is an invariant of a different character than the Vassiliev invariants derived here. However, this calculation allows for the possibility that the Vassiliev invariants and the invariants associated to this averaging procedure both have an interpretation in terms of the expansion of the polynomial invariant. The expansion of the average invariant begins with terms of 0th and 2nd order while the Vassiliev invariant begins at the 1st order. The two sequences could live at different orders in the expansion of the polynomial invariant. The average method of vertex

\(^7\) This may be derived by requiring that the intersection \( \times \), built from \( \times \) and \( \times \), is compatible with both the Mandelstam identities and the Kauffman bracket skein relations (Eq. (A.1)). It might be possible to generalize this construction for higher valence vertices of oriented graphs.
4.2. Trivalent vertices

While it is not possible to decompose odd valence vertices into over and under crossings in the same manner as the last section, it is possible to learn about the invariants and, in particular, framing. Generically, a diffeomorphism with a non-vanishing volume factor will give a non-zero contribution to the variation. Take, for example, the diffeomorphism which rotates the trivalent vertex shown in Fig. 4. One may begin with a twisted 3-vertex and rotate through an angle of $2\pi$. It is easy to see that the effect of this diffeomorphism is to cross the edges $e_1$ and $e_2$. Making use of the transverse 4-vertex decomposition,

\begin{equation}
\langle \begin{array}{c}
\text{a} \\
\text{b} \\
\text{c}
\end{array} \rangle - \langle \begin{array}{c}
\text{a} \\
\text{b} \\
\text{c}
\end{array} \rangle = \frac{\pi i}{k} \sum_{i=\max(a-b)}^{a+b} (-1)^{a_i(a,a)} a_i(a,b) \frac{\lambda_i^{ab} \Delta_i}{\theta(a,b,i)} \langle \begin{array}{c}
\text{a} \\
\text{b} \\
\text{c}
\end{array} \rangle
\end{equation}

in which the identity of Eq. (A.9) is used in the second line. This result is exactly what is expected from the first-order expansion of the $\lambda$-move (Eq. (A.18))! It is easy to see that the classical group recoupling determines the first-order result. What is more, the operator is diagonal on this vertex. The eigenvalue may be then exponentiated to recover the whole series

\begin{equation}
\langle \begin{array}{c}
\text{a} \\
\text{b} \\
\text{c}
\end{array} \rangle = \lambda^{ab} \langle \begin{array}{c}
\text{a} \\
\text{b} \\
\text{c}
\end{array} \rangle
\end{equation}

with

\begin{equation}
\lambda = (-1)^{(a+b-c)/2} A^{[a(a+2)+b(b+2)-c(c+2)]/2} \langle \begin{array}{c}
\text{a} \\
\text{b} \\
\text{c}
\end{array} \rangle.
\end{equation}

The vertex is not completely simple, though. The tangent space structure affects the result strongly. Under general variation the 3-vertex is
\[
\int_{-\epsilon}^{\epsilon} \frac{du}{du} \langle U_{\nu} \rangle = \frac{\pi i}{2k} \sum_i \Delta_i \\
\times \left[ (\kappa(1, 3) + \kappa(3, 1)) \sum_i (-1)^{a_i(a,c)} \frac{a_i(a,c)}{\theta(a,c,i)} \left\langle \begin{array}{c}
\text{Diagram 1}
\end{array} \right\rangle \right] \\
+ (\kappa(2, 3) + \kappa(3, 2)) \sum_i (-1)^{a_i(b,c)} \frac{a_i(b,c)}{\theta(b,c,i)} \left\langle \begin{array}{c}
\text{Diagram 2}
\end{array} \right\rangle \\
+ (\kappa(1, 2) + \kappa(2, 1)) \sum_i (-1)^{a_i(a,b)} \frac{a_i(a,b)}{\theta(a,b,i)} \left\langle \begin{array}{c}
\text{Diagram 3}
\end{array} \right\rangle.
\]

To reproduce the results derived above one can consider decompositions which have, for some value of \( u \), an intersection between the edges \( e_1 \) and \( e_2 \) (and which do not move the edge \( e_3 \)).

For the rotation shown in Fig. 4b the edge \( e_3 \) remains fixed. Thus \( \theta_3^2 = 0 \). With the clockwise convention shown in the figure, this reduces to simply \(^8\)

\[
\int_{0}^{\pi} \frac{du}{du} \langle U_{1u} U_{2u} U_{3u} \rangle = -\frac{\pi i}{2k} \left[ (-1)^{a_i(a,b)} a_c(a,b) + (-1)^{a_i(b,c)} a_a(b,c) \right] \left\langle \begin{array}{c}
\text{Diagram 4}
\end{array} \right\rangle.
\]

This calculation lacks the explicit projection dependence of the first calculation in that the variation is carried out with a diffeomorphism in the three-manifold, without reference to any preferred direction. In fact, the result of the diffeomorphism may be entirely different. For instance, if the first and third edges were part of a single line so that \( \theta_3^2 = \theta_3^q \), as in Fig. 4c, then this diffeomorphism would only rotate the second edge. The recoupling coefficient would be

\[
\frac{\pi i}{2k} \kappa(2, 1) (-1)^{a_i(b,c)} b(b + 2)
\]

- quite different from the previous result! This is another example of how the tangent space structure determines the invariant. The usual \( \lambda \)-move is only recovered for non-collinear, essentially planar graphs with a global projection. (By essentially planar diagrams I mean projected graphs with intersections created in the projection labeled with "over" and "under" crossings.) This is expected for Temperley–Lieb recoupling theory and the Kauffman bracket as these are invariants for graph, knot, and link diagrams. In the next section, this becomes quite clear when single lines are analyzed.

\(^8\) Here, the volume factors have been "regulated" so that each factor has the same value in the limit when the edge parameter for \( e_1 \) (and \( e_2 \)) vanishes as its value for any finite parameter.
4.3. Single line framing

Framing was introduced to cure the ambiguities in the volume factor for knots. When studying invariants of spin nets, framing must be invariably studied on single edges. After recovering the result of Temperley–Lieb recoupling theory, I will examine the ambiguity of the volume term in more detail. For an embedded graph, there are several options, framing cycles in the graph, reaching further into tangent space, and balancing with another network as in the Yetter–Barret–Crane spin network invariants [33].

Curls may be treated as an application of the 4-vertex decomposition. In blackboard framing, since twists are projected as curls, the number of curls in a knot diagram is equivalent to the self-linking number. This number may be changed with a decomposition parameter. I take the parameter \( u \) to interpolate between curls of different chirality, i.e. as \( u \) flows from positive to negative, the positive curl \( \smile \) changes into a negative one \( \frown \). The intersection forms at \( u = 0 \). This change in the line is easily captured by the transversal four-valent vertex decomposition.

The variation of the parameter \( u \) results in

\[
\int_{-\epsilon}^{+\epsilon} \frac{du}{du} (U_e) = \left\langle \smile \right\rangle - \left\langle \frown \right\rangle
\]

\[
= \frac{\pi i}{k} \sum_c (-1)^a a_c(a,a) a_c(a,a) \frac{\Delta_c}{\theta(a,a,c)} \left\langle \begin{array}{c} \smile \\ c \end{array} \right\rangle
\]

\[
= \frac{\pi i}{k} (-1)^a \sum_c (-1)^c \left( a(a + 2) - \frac{c(c + 2)}{2} \right) \frac{\Delta_c}{\theta(a,a,c)} \left\langle \begin{array}{c} \begin{array}{c} \smile \\ c \end{array} \\ a \end{array} \right\rangle.
\]

(23)

The identity of Eq. (A.19) was used in the second line.

There is a new feature in the calculation. To relate a curl to a uncurled line, one must shrink the loop to a point. One might try to perform this transformation with a second parameter much as was done for the six-valent vertex. However, the volume factor vanishes. (The volume vanishes for both the first line, no intersections, and for the last line, a planar deformation, of Eq. (23).) One must merely collect the group factors. Making use of the identities of Eqs. (A.9), (A.6), and (A.7),

\[
\left\langle \smile \right\rangle - \left\langle \frown \right\rangle = \frac{\pi i}{k} (-1)^a \sum_c (-1)^c \frac{\Delta_c}{\Delta_a} (a(a + 2) - c(c + 2)/2) \left\langle s \right\rangle
\]

\[
= (-1)^a a(a + 2) \frac{\pi i}{k} \left\langle s \right\rangle.
\]

(24)

This is the expected result from Temperley–Lieb recoupling theory,

\[
\left\langle \smile \right\rangle = (-1)^a A^{a(a+2)} \left\langle s \right\rangle.
\]
Again, this holds only for blackboard framing.

This is a special case. One could also perform a twist on a single line. Since the boundary values of the edges must be left unchanged, the edge may only be rotated by even multiples of $\pi$. Rotating one end $2\pi$ with respect to the other gives

$$\int_0^{2\pi} du \frac{d}{du} \langle U_e \rangle = \frac{4\pi i}{k} \int du \int dt \int ds \, \epsilon_{abc} \dot{e}^a(t) \dot{e}^b(t) \dot{e}^c(s) \delta^3(e(t), e(s)) \times \langle U_e(0, s) T^a_{(a)} U(s, t) T^a_{(a)} U(t, 1) \rangle.$$ 

While the recoupling gives an overall factor of $(-1)^a \frac{\pi i}{k} a(a+2)$, the volume factor is clearly ill-defined. Without self-intersections in the region lifted by $u$, there are only the one-dimensional solutions $s = t$. Usually one regulates this volume factor by displacing one of the edges. By so changing one of the tangents in the volume factor of Eq. (9), the delta-function may become well defined. For loops this becomes the linking number of the loop and its frame; to account for this framing one may associate an integer to each component of a link. In the approach taken in this paper, in which subgraphs with vertices are analyzed, the framing is effectively used only in a small region. As the self-linking of edges is meaningless, the usual approach can only be recovered if the frame is “matched” at the boundary.

Alternately, one may regulate the open edge volume factor by reaching deeper into tangent space. Introducing a background metric and expanding one of the tangents, one finds

$$(-1)^a \frac{\pi i}{k} a(a+2) \int dt \, \epsilon_{abc} \left. \frac{\dot{e}^a(t)}{\dot{e}(t)} \right|_{t=0} \dot{e}^b(t) \dot{e}^c(t) \frac{1}{|\dot{e}(t)|^3}$$

which may be interpreted as an infinitesimal form of the self-linking number.

Another point of view is suggested by the definition of a holonomy. The path ordered exponential may be given by the limit

$$\mathcal{P} \exp \left[ - \int_0^1 dt \, \dot{\alpha}^a(t) A_a(\alpha(t)) \right] = \lim_{N \to \infty} \prod_{i=1}^N \left( 1 + A_a d\alpha^a \right).$$

The loop $\alpha$ only becomes smooth in the limit. Since it is expected that space is only effectively continuous at a macroscopic scale, the holonomies ought to be constructed from Planck length segments and are only defined for large but finite $N$. In this case, the effective volume factors are a set of paired terms like those given in Eq. (4). Since these come in pairs which only differ by sign, they cancel. In this interpretation, the difficulty of single line framing is an artifact of the use of a smooth manifold. It would be useful to see whether the framing difficulty remains when calculated on other manifolds.

As final observation of single-line framing, it is interesting to note that the volume ambiguity may be handled in yet another way. Instead of regulating the delta-function and leaving the theory with an arbitrary element, one may instead ensure that this element cancels with another similar term. The invariant then becomes ambient isotopic
rather than regular isotopic. To see how this may be accomplished, consider a path $\alpha$ and its associated frame $\alpha_f$. The framed path is related to its partner by a direction field $\theta$, $\alpha_f = \alpha + \eta \theta$. (To capture framing information one is really interested in the equivalence class of smooth deformations along the loop of such direction fields. However, this will not affect the calculation here.) The effect is, of course, to consider two copies of the graph, one "slightly displaced" from the other. Typically, one frames a loop in this manner, calculates the expectation value and then takes the limit $\eta \to 0$. The result is taken to be the expectation value of the original path. Here, the variational method is used to see how it is possible to remove the framing dependence.

Under a variation of a framed path both the path and its frame are parameterized by $u$. The variation has a total of four terms,

$$\frac{d}{du} \langle U_\alpha U_{\alpha_f} \rangle = \frac{4\pi i}{k} \int dt \int ds \epsilon_{abc} \hat{\alpha}^a(s) \hat{\alpha}^b(s) \hat{\alpha}^c(t) \delta^3(\alpha(s), \alpha(t))$$

$$\times \langle U_\alpha (0,s) T^i U_\alpha (s,t) T^i U_\alpha (t,1) U_{\alpha_f} \rangle$$

$$+ \int dt \int ds \epsilon_{abc} \hat{\alpha}^a(s) \hat{\alpha}^b(s) \hat{\alpha}^c(t) \delta^3(\alpha(s), \alpha_f(t))$$

$$\times \langle U_\alpha (0,s) T^i U_\alpha (s,1) U_{\alpha_f} (0,t) T^j U_{\alpha_f} (t,1) \rangle$$

$$+ \int dt \int ds \epsilon_{abc} \hat{\alpha}^a(s) \hat{\alpha}^b(s) \hat{\alpha}^c(t) \delta^3(\alpha_f(s), \alpha(t))$$

$$\times \langle U_\alpha (0,t) T^j U_\alpha (t,1) U_{\alpha_f} (0,s) T^i U_{\alpha_f} (s,1) \rangle$$

$$+ \int dt \int ds \epsilon_{abc} \hat{\alpha}^a(s) \hat{\alpha}^b(s) \hat{\alpha}^c(t) \delta^3(\alpha_f(s), \alpha_f(t))$$

$$\times \langle U_\alpha U_{\alpha_f} (0,s) T^j U_{\alpha_f} (s,t) T^i U_{\alpha_f} (t,1) \rangle \right].$$

(25)

Two terms have insertions in only one path, while the other two terms have insertions in both. I have omitted two terms like the first and last lines, which have $s > t$. With an eye to the limit $\eta \to 0$, it is reasonable to take $\hat{\alpha} = \hat{\alpha}_f$; the decomposition parameter derivatives are in the same direction. This condition ensures that the second and third terms of Eq. (25) differ by a sign and cancel. Expanding the remaining terms, the framed path $\alpha_f$ in terms of the base loop $\alpha$ and the frame field $\theta$, one finds that the terms linear in $\eta$ cancel and the last equation reduces to

$$\frac{d}{du} \langle U_\alpha U_{\alpha_f} \rangle = \frac{4\pi i}{k} \int dt \int ds \epsilon_{abc} \hat{\alpha}^a(s) \hat{\alpha}^b(s) \hat{\alpha}^c(t) \left[ \delta^3(\alpha(s), \alpha(t))

\times \langle U_\alpha (0,s) T^i U_\alpha (s,t) T^i U_\alpha (t,1) U_{\alpha_f} \rangle

+ \delta^3(\alpha_f(s), \alpha_f(t)) \langle U_\alpha U_{\alpha_f} (0,s) T^j U_{\alpha_f} (s,t) T^i U_{\alpha_f} (t,1) \rangle \right].$$

(26)

Despite the remaining ambiguity in the volume factor, the result of the variation can
vanish if the recoupling on the two edges differs by a sign. This opens up a number of possibilities. One could simply insert an $i$ in the definition of the holonomy. This unfortunately means that the holonomy is no longer gauge invariant in that it no longer satisfies the Gauss constraint of canonical quantum gravity [14]. A far more elegant solution is to require that the variation of two identically labeled $SU(2)$ networks differs by a sign. Since these results are first order, it suggests that one need only change the overall sign of the action so that $e^{iS}$ goes to $e^{-iS}$. The first-order coefficient is, in the full series, exponentiated to the complex quantity $q$, so this sign change is a matter of taking the complex conjugate of the variable of the polynomial invariant. This is the balanced $SU(2) \times SU(2)$ invariant of Barrett–Crane [20] and Yetter [33].

One might also try to build a balanced $SU(2) \times U(1)$ network. As was noticed some time ago in the context of simple loops (see, for instance, Ref. [34]), the frame on a $SU(2)$ knot can be balanced with a frame on an identical $U(1)$ knot so that the complete polynomial invariant does not depend on the frame. To see how this might arise, consider a $SU(2) \times U(1)$ Chern–Simons theory. The action for the $U(1)$ part is simply

$$s(a) = \frac{k'}{4\pi} \int d^3x \epsilon^{abc} \partial_a a_b a_c$$

in which $a_a$ is the $U(1)$ connection. The action of the composite theory with the connection $A^B_A = A^i(T^i)^B_A + \alpha \delta^B_A$ has the same form as the $SU(2)$ action of Eq. (1) [35]. Under variation the $U(1)$ part of the theory is identical except for the group structure. For instance, the curl relation is

$$\langle \left\langle \begin{array}{c} 0 \\ U(1) \end{array} \right\rangle \rangle_{U(1)} - \langle \left\langle \begin{array}{c} 0 \\ U(1) \end{array} \right\rangle \rangle_{U(1)} = (-1)^d e^{2\pi i/k} \langle \left\langle 1 \right\rangle \rangle_{U(1)}.$$  

Clearly, if the total theory is to be frame independent, the expectation values based on the two parts of the theory must be balanced [33]. It seems that this may be accomplished in general only if one is willing to assign irrational charges to the $U(1)$ edges; in the general case one must require that the label on the abelian network be $\sqrt{n(n+2)}$, where $n$ is the label on the corresponding $SU(2)$ network.

### 4.4. Higher valence vertices

For higher valent vertices the same recoupling formula of Eq. (18) may be used iteratively to find the action of the gauge invariant operators in the spin net basis. However, there does not seem to be a canonical form of the intertwiners which form an eigenspace of the operators. Nevertheless, the recoupling procedure can be applied to the higher order decompositions of Eq. (16).

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9 If the recoupling identity used in calculating the area operator eigenvalues in the diagrammatic approach could be generalized for the coefficients $a_s(a,b)$, it would be easy to find a suitable intertwiner tree. Alternately, there may be a relation among the spin operators at the vertex which suggests an intertwiner basis. (See Ref. [16].)
Despite this, there is a class of graphs which have a particularly simple decomposition. These graphs have edges in the fundamental representation (labeled by 1) and even-valence vertices which are "consistently oriented." That is, all the vertices have incident edges which are oriented in alternating directions as projected along the vector field associated with the decomposition (as in Fig (2)). For these graphs, the evaluation of the operators, making use of the identity of Eq. (17), yields simple connection diagrams; the variation relates the decomposition of the vertex into over and under crossings to a sum of links with non-intersecting components. Assuming it is possible to "smooth" the edges to remove the kinks at the vertices, the effect of the operators then simply produce a sum of regular link invariants. In these cases, the Vassiliev invariant is given by the chromatic evaluation of the loops.

5. Discussion

In this paper new invariants for embedded graphs in Chern–Simons theory. The key difference from earlier work is that the new invariants depend on the tangent space structure at vertices. Using the variational technique, this analysis of the definition of the vacuum expectation value of embedded graphs or, equivalently, of the spin net representation of the Kodama state, suggests that one may sensibly define invariants of graphs. For closed graphs, these operators are simply a sum of signed terms consisting of Wilson graphs with generators inserted at the vertices. The result is a Vassiliev invariant of order \((n - 1)\) for a \(2n\)-valence vertex and so provides an example of the theorem of Birman and Lin on the expansion of a polynomial invariant. The beauty of the variational method of graph invariants lies in that it creates relations between different non-intersecting invariants of the 3-manifolds without resorting to a fixed background structure. The result is an invariant which depends not only on the 3-manifold but also the tangent space at the vertices. Graph invariants then capture information of the three manifold as well as the tangent space at the vertices. In the spin net basis it is possible to evaluate the action of these operators. Some of these variational operators may be formally exponentiated to give the result to all orders. In this manner, the Temperley–Lieb recoupling theory of Kauffman and Lins is recovered for essentially planar diagrams with blackboard framing. The calculation reveals that it is not necessary to separately frame vertices as the variation is well-defined without further regulation. By examining this perspective the framing of a single edge of a spin network, the balanced spin network invariants of Barrett–Crane–Yetter were recovered [20,33].

Motivated by considerations arising from canonical quantum gravity, this study focused on the role of framing at the vertices. Though spin net geometry does not require framing to be rigorously well defined, there are three immediate reasons why one might expect that framing is a property required by the full theory. First, the cosmological constant appears in the invariant. Since the cosmological constant appears only in the Hamiltonian constraint, framing is an issue of dynamics. While it may be that framing is only required for the "Kodama phase" of the theory, the state suggests that the complete
theory (taking seriously the suggestion that the Kodama state is a well-defined solution of the full theory) will need to account for framing in at least one sector. Second, in the loop representation of the linear theory [36] as well as Maxwell theory [37], framing plays a key role. Third, since framed links are sufficient to construct all compact, oriented 3-manifolds [38], a theory of the dynamics of such manifolds ought to have a framed loop representation.

Even so, the Kodama state provides an enigma for canonical quantum gravity. While it has all the expected characteristics and is a formal solution to the constraints, it cannot be treated with the theory of measures which has proven so fruitful in spin net geometry [11]. A key conceptual confusion has been the lack of understanding of the requirement of diffeomorphism invariance. Is the classical three-manifold diffeomorphism invariance broken by dynamics? In the language of knot theory, are kinematic states regular or ambient isotopy invariants? While it seems obvious that when dealing with such a canonically diffeomorphism invariant theory as gravity that we must consider invariants of ambient isotopy, such as the Jones Polynomial, it is only clear that in the classical limit gravity possesses the full diffeomorphism invariance. Indeed, given the universal nature of Chern–Simons action, it even seems possible that this state may be the scaling limit of an underlying theory of quantum gravity. It perhaps indicates that a more sensitive invariant may be required to describe the full theory. One can only hope that it is pointing to a feature of the microstructure of space-time.

On a more prosaic level, the representation of framed spin networks created in part to describe states in the Kodama phase [27], offers a number of simple mathematical challenges: What is the product of two framed loops which intersect? Is such a product unique? What is the appropriate algebra for the basic operators of the theory? The variational and recoupling techniques offer a method for resolving these issues. For these reasons, this study concentrated on the issues of frame and decomposition of embedded graphs.

The results suggest that it is possible to further and consistently define the vacuum expectation value of a graph. Through a delicate interplay of group and tangent space structure, the invariant seems to be defined. It is reasonable to expect that framed graphs include tangent space information in addition to that of the three manifold. As is suggested by geometric operators in spin network kinematics, the tangent space plays a large role. The invariants of framed graph depend critically on the tangents of the incident edges. This work offers a hint of how tangent space and manifold information might be folded together. One potentially interesting direction to explore is the evaluation for flat connection. Since the holonomies reduce to identity, the result would be a numerical knot invariant [22]. Composed of group, manifold, and tangent space structure, these invariants would give some intuition of spin net expectation values as well as the state space of spin net geometry.

I would like to close with two additional comments. The first is a bit technical. As noted in Section 3, the form of the vertex decomposition is given by the choice of the decomposition vectors at the vertices $\epsilon_{\mu\nu\rho}$. These vectors determine both the signs arising from the volume factor and which non-intersecting links appear. The freedom
in the choice of these vectors is the freedom in the definition of the vertex [32]. One way to identify this freedom is by identifying the “tangent space volume” in which the decomposition vector lies [32]. This is based on the observation that a diffeomorphism acts as an invertible linear transformation on the incident edge tangents. A volume, identified by a non-co-planar triple, cannot be made to vanish by a diffeomorphism. This suggests that the amount of freedom in defining a vertex is contained in how the graph is embedded in the manifold.

The second comment is more speculative. Since there is no natural relation between the decomposition vectors at distinct vertices, the graph invariant seems to be ambiguously defined without a global projection. If the expectation values of spin nets are invariants of the 3-manifold rather than invariants of planar diagrams, then there must be a consistent definition. As the expectation value may differ by framing and as the framing dependence collects into an overall factor, I would like to suggest a gauge principle for the framing of embedded graph invariants: it ought to be possible to choose any decomposition of each distinct vertex in a graph and still find the same invariant - up to “gauge.” There is a hint of this in the planar representation of framed graphs. One simple example is of two subgraphs connected by a single edge. In a global projection, distinct decompositions of the graph ought to be identical up to gauge. For instance, if one subgraph is rotated $2\pi$ with respect to the other, the result is, of course, a curl. This may be the phase of an abelian gauge theory. Another simple example is given by a pair of trivalent vertices joined by a pair of edges. In a global projection, the decomposition of the graph ought to be identical, up to gauge, if one vertex is rotated $2\pi$ about an axis joining the two vertices. A simple application of recoupling theory shows that the effect of the rotation is simply equal to a curl on one of the “external lines.” This again suggests a curl as a “phase” due to a gauge rotation. While it is not clear from these examples whether such a gauge principle would hold on arbitrary vertices and arbitrary decomposition vectors, it nonetheless suggests that the framing dependence of vacuum expectation values of embedded graphs in Chern–Simons theory is gauge.

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10 This idea was first mentioned in a discussion with Lee Smolin, Roumen Borissov and myself at the Center for Geometry and Gravitational Physics.
Appendix A

This appendix contains the basic definitions and formula of recoupling theory. While this work uses for the most part the binor conventions for SU(2) recoupling theory, the formula here are for more general $A$ (except where explicitly stated). For more than a brief review see Ref. [19]. The complex phase $A$ is given by

$$A = e^{i\pi/2k}$$

for integer $k$. $A$ is found in the fundamental skein relation for the Kauffman bracket

$$\langle a \rangle = A \langle a \rangle + A^{-1} \langle a \rangle^{-1}$$

(A.1)

and is related to the usual parameter $q$ via

$$q = A^2 = e^{i\pi/k}.$$ 

In, $q$-quantum gravity, this parameter is given by

$$q = \exp\left(\frac{i A l_P^2}{6}\right)$$

so that $k = 6\pi / A l_P^2$ (an integer!).\(^{11}\) The "classical" or "binor" limit occurs when $q = 1$ and $A = -1$ ($r \to \infty$) so that $\hbar$ and/or $\Lambda$ vanish. In this limit, the relation of Eq. (A.1) (which is also the relation among $2 \times 2$ matrices known as the Mandelstam identities) take the simple diagrammatic form of

$$\langle a \rangle + \langle a \rangle^{-1} + \langle a \rangle^{-1} = 0.$$ (A.2)

Recoupling theory begins with the basic irreducible representation, diagrammatically a single line or "strand." Closing this line (or taking its trace) gives the loop value $d = -A^2 - A^{-2}$. The classical value (when $A = \pm 1$) of $d$ is $-2$. Higher representations may be built from the basic line using the Wenzel–Jones projector defined by

$$\frac{1}{n} = \frac{1}{n!} \sum_{\sigma \in S_n} (A^{-3})^{[\sigma]} \circ \circ$$

(A.3)

in which the sum is over elements of the symmetric group, $\sigma$; $|\sigma|$ is the sign of permutation; the expansion $\circ \circ$ is given in terms of the positive braid (the strands are only over crossed $\times$); and the asymmetric quantum number $\{n\}$ is defined by

\(^{11}\) If one includes $CP$-breaking term in the action, $\int F \wedge F$, and the non-perturbative renormalization parameter [3], then one has $k \to k + 2$ with

$$k = \frac{6\pi}{A l_P^2} + \alpha.$$ 

The parameter $\alpha$ is the phase coming from the $CP$-breaking term.
This quantum integer is simply an integer in the classical limit.

The evaluation of a single un-knotted $n$ loop is

$$\Delta_n \equiv (-1)^n [n + 1], \quad (A.5)$$

where $[n + 1]$ is the dimension of the representation and the brackets identify the symmetric quantum integer defined by

$$[n] = \frac{A^{2n} - A^{-2n}}{A^2 - A^{-2}}.$$

For classical spin networks $A = -1$, the next two identities are useful in the calculation of the curl

$$\sum_{c=0, \text{even}}^{2a} (-1)^c \Delta_c = \Delta_a \quad (A.6)$$

and

$$\frac{(-1)^a}{\Delta_a} \sum_{c=0, \text{even}}^{2a} (-1)^c \frac{c(c + 2) \Delta_c}{4} = a(a + 2). \quad (A.7)$$

The function $\theta(a, b, n)$ is given by

$$\theta(m, n, l) = \frac{(-1)^{(a+b+c)}[a+b+c+1][a][b][c]}{[a+b][b+c][a+c]}, \quad (A.8)$$

where $a = (l+m-n)/2$, $b = (m+n-l)/2$, and $c = (n+l-m)/2$. A "bubble" diagram is proportional to a single edge,

$$\frac{-1}{n+1} \theta(a, b, n) \quad (A.9)$$

The basic relation relates the different ways in which three angular momenta, say $a$, $b$, and $c$, can couple to form a fourth one, $d$. The two possible recouplings are related by the formula

$$\sum_{|a-b| \leq i \leq (a+b)} \{a \ b \ i \ c \ d \ i'\} = q_{6j} \quad (A.10)$$

where the $q_{6j}$-symbol on the right-hand side is defined below. It is closely related to the Tet symbol. Variously, drawn as
it is defined by \[19\]
\[
\begin{pmatrix}
  b & f \\
  a & d
\end{pmatrix} = \text{Tet} \begin{pmatrix}
  a & b & e \\
  c & d & f
\end{pmatrix},
\]
(A.11)
\[
\text{Tet} \begin{pmatrix}
  a & b & e \\
  c & d & f
\end{pmatrix} = N \sum_{m \leq s \leq S} (-1)^s \frac{[s + 1]!}{\prod_i [s - a_i]! \prod_j [b_j - s]!}
\]
(A.12)
\[
N = \frac{\prod_{i,j} [b_j - a_i]!}{[a]![b]![c]![d]![e]![f]!}
\]
(A.13)
in which
\[
\begin{align*}
  a_1 &= \frac{1}{2}(a + d + e), & b_1 &= \frac{1}{2}(b + d + e + f), \\
  a_2 &= \frac{1}{2}(b + c + e), & b_2 &= \frac{1}{2}(a + c + e + f), \\
  a_3 &= \frac{1}{2}(a + b + f), & b_3 &= \frac{1}{2}(a + b + c + d), \\
  a_4 &= \frac{1}{2}(c + d + f), & m &= \max \{a_i\}, & M &= \min \{b_j\}. \\
\end{align*}
\]
(A.14)
The \(q6j\)-symbol is then defined as
\[
\begin{pmatrix}
  a & b & i \\
  c & d & j
\end{pmatrix} = \text{Tet} \begin{pmatrix}
  a & b & i \\
  c & d & j
\end{pmatrix} \Delta_i
\]
\[
\frac{\theta(a, d, i) \theta(b, c, i)}{\theta(a, d, i) \theta(b, c, i)}.
\]
(A.15)
These satisfy a number of properties including the orthogonal identity
\[
\sum_l \begin{pmatrix}
  a & b & l \\
  c & d & j
\end{pmatrix} \begin{pmatrix}
  d & a & i \\
  b & c & l
\end{pmatrix} = \delta_i^j
\]
(A.16)
and the Biedenharn–Elliott or Pentagon identity
\[
\sum_l \begin{pmatrix}
  d & i & l \\
  e & m & c
\end{pmatrix} \begin{pmatrix}
  a & b & f \\
  e & l & i
\end{pmatrix} \begin{pmatrix}
  a & f & k \\
  c & d & l
\end{pmatrix} = \begin{pmatrix}
  a & b & k \\
  c & d & i
\end{pmatrix} \begin{pmatrix}
  k & b & f \\
  e & m & c
\end{pmatrix}.
\]
(A.17)
The “\(A\)-move”
\[
\begin{pmatrix}
  e \\
  \text{a}
\end{pmatrix} = \lambda^{ab}_{c}
\]
(A.18)
where
\[
\lambda^{ab}_{c} = (-1)^{(a+b-c)/2} A^{a(a+2)+b(b+2)-c(c+2)}/2.
\]
An over-crossing may be related to a recoupling via
\[
\begin{pmatrix}
  e \\
  \text{a}
\end{pmatrix} = \sum_{i=[a-b]}^{a+b} \lambda^{ab}_{i} \frac{\Delta_i}{\theta(a, b, i)} \begin{pmatrix}
  \text{a} \\
  \text{b}
\end{pmatrix}.
\]
(A.19)
A similar identity holds for under-crossings. In the binor limit the two recouplings coincide.

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