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On the Quantization of Plane Gravitational Waves in Loop Quantum Gravity

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Abstract

Gravitational waves are oscillations in the geometry of spacetime caused by the movement of mass or energy. In loop quantum gravity, geometry is quantized so spatial quantities such as area and volume take on discrete values. We worked on quantizing plane gravitational waves, which propagate in one direction and have uniform wavefronts perpendicular to the direction of travel, using the method of canonical quantization on general relativity.

For canonical quantization, a theory is expressed in terms of its canonical variables and constraints on those variables, then the variables and constraints are taken to quantum operators and the states of the quantum theory are those that satisfy the constraint operators. The canonical variables of general relativity are Ashtekar variables, which represent gravitational fields in a way that reveals their similarity to electromagnetic fields and other gauge fields. The symmetries of general relativity are formulated as constraints. We used vectors to visually represent the constraints on a spacetime diagram in a manner that is consistent with their actions and their algebra. We represented the constraints as either a Lorentz boost, a directional shift, or a propagation along a light cone. We verified the algebra of these constraints for gravitational waves. A correct quantum theory will have algebra that is consistent the classical constraint algebra.

We found possible quantum operators for the unidirectional constraint, the constraint that ensures the plane wave moves only in one direction. We found quantum states satisfying the different versions of the operator and checked that these states were normalizable. We determined that two different versions of the operator had nontrivial normalizable solutions. These solutions could be used in conjunction with a consistent quantization of the other constraints as a quantum formulation of plane gravitational waves.

Acknowledgements

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Contents

Introduction

One of the most glaring gaps in our knowledge of physics is the problem of how to combine general relativity and quantum mechanics into a single theory. The Standard Model provides a quantum mechanical theory for all of physics except for general relativity, necessitating a theory of quantum gravity. This theory would yield Einstein's theory of general relativity in the classical limit but would also correctly account for quantum effects on smaller scales. The hope is that a theory of quantum gravity would provide descriptions of phenomena that general relativity cannot fully describe such as Hawking radiation, the singularity at the center of a black hole, and the very early universe.

The biggest challenge to any theory of quantum gravity is the complete lack of experimental evidence of quantum gravity effects. This lack of evidence is understandable given the size of Planck length, which is the approximate scale at which quantum gravity effects dominate. The Planck length ℓ_p is found by combining the speed of light c, the reduced Planck constant $\hbar$, and Newton's gravitational constant G in the simplest way that yields units of length. It is

$$\ell_p = \sqrt{\frac{\hbar G}{c^3}} \approx 1.6 \cdot 10^{-35}\text{m}. \tag{1}$$

Despite the size of the Planck length, there is a possibility of observing quantum gravity effect that originate on the Planck length but have a larger effect.

The Planck length is generally considered to be the smallest unit of length and the Planck volume ℓ_p^3 is roughly the smallest unit of volume. One approach to quantum gravity posits that this smallest unit of volume is a quantum of volume and that space is a discrete entity made up of quanta of volume. This approach characterizes loop quantum gravity. In loop quantum gravity, states of spatial geometry are represented as a three-dimensional labeled graph[1] called a spin network. A spin network can also be thought of as a net of intersecting loops with vertices where the loops touch, which gives the theory its name [19]. Quantum operators acting on a spin network return observables in terms of the labels. The eigenvalues of the volume operator are in terms of the labels associated with the vertices of a spin network and the eigenvalues of the area operator are in terms of the label associated with the edges.

To get a better of what quantized space is, consider a particle in a one-dimensional quantum harmonic oscillator. Recall that the eigenstates have energies $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$ where $n = 0, 1, 2, \ldots$. Then the ground state energy is $E_0 = \frac{1}{2}\hbar\omega$ and the energy of every excited state can be expressed in terms of E_0. For example, $E_1 = 3E_0$ and $E_7 = 15E_0$. The quantity E_0 is considered a quantum of energy. Just as the excited states of a particle in a box possess quanta of energy, space is made up of quanta of volume, roughly the size of the Planck volume. No measurement could ever yield a smaller volume in the same way that no measurement of the energy of a particle could yield a value below the energy of the ground state. Similarly, other spatial quantities such as angles and areas are made of quanta.

To create a quantum operator in loop quantum gravity, one starts with a classical quantity and uses a method called canonical quantization to change the classical quantity into an operator. Loop quantum gravity starts with classical general relativity expressed in Ashtekar variables. Ashtekar variables must satisfy a set of constraints in order to account for the symmetries of general relativity, so these constraints must be quantized. The constraints are the Gauss constraint, the diffeomorphism constraint, and the Hamiltonian constraint, which ensure invariance under rotation, diffeomorphism, and spacetime translation, respectively. The unidirectional constraint is an additional constraint that forms plane gravitational wave when satisfied

[1]In this sense, a graph is a set of vertices along with a set of edges.

[13]. Our work focuses on creating a quantum operator from the classical unidirectional constraint and on finding states that satisfy the operator.

One motivation for undertaking this project is the possibility of insight into modified dispersion relations for gravitational waves. Modified dispersion relations take $E^2 = p^2 + m^2$ to be an approximation where the term p^2 is the first term of a Taylor series expansion. The modified energy-momentum relation is

$$E^2 = p^2 + m^2 + \kappa \frac{p^3}{M_p} + \dots \tag{2}$$

where M_p is the modified Planck mass given by

$$M_p = \sqrt{\frac{\hbar c}{8\pi G}} \approx 2.4 \cdot 10^{18} \text{GeV}/c^2 \tag{3}$$

which, like the Planck length, is the simplest combination of constants yielding the correct units. The factor of $\sqrt{\frac{1}{8\pi}}$ is included simply for computational convenience. Given the size of the Planck mass, the effects of the modification are small but detectable at high momentum because the modification breaks local Lorentz invariance. Effects that could be detected experimentally include new decay channels, shifts in current threshold energies, and new upper thresholds. The modification term on the order of p^3 has been experimentally ruled out, but the possibility of terms on the order of p^4 or higher remains. Since modified dispersion relations govern detectable effects originating on the Planck scale, they provide a unique link between quantum gravity and experiment. If modified dispersion relations were to arise from this work, they could guide the quantization process.

Part 1 deals exclusively with classical general relativity. We begin by introducing general relativity and gravitational waves in their standard variables in Sections 1 and 2. We then introduce canonical variables using the canonical variables of electromagnetism in Sections 3 and 4 as an example to give the reader some background to understand Ashtekar variables. We give the constraints that Ashtekar variables must satisfy and show a way to represent them on spacetime diagrams in Sections 5 to 7. The representation of the constraints on spacetime diagrams is original work.

In Part 2, we transition from the classical regime to the quantum regime using canonical quantization of general relativity expressed in Ashtekar variables to construct the spin networks of loop quantum gravity. In Section 8, we again use an example from electromagnetism, the Aharonov-Bohm effect, to understand how to deal with gauge dependent canonical variables, which motivates the loop structure of loop quantum gravity. We explain the method of canonical quantization in Section 9 and then elaborate further on the mathematics of loop quantum gravity in Section 10.

Finally, in Part 3, we present our original work. We use canonical quantization to create multiple quantum operators from the unidirectional constraint. We outline the preliminaries of quantization in Section 13. Four different unidirectional operators are given in Sections 14 to 17. In Sections 14 and 15, we find solutions that satisfy the operators and determine the conditions for these solutions to be normalizable. We discover two nontrivial normalizable solutions. These solutions are significant because they represent plane gravitational waves within quantized space.

Note that throughout this thesis, we set $c = 1$, essentially making meters and seconds into the same unit. This choice is often made in theoretical physics with the intention of making the calculations cleaner, and the missing factors of c may be recovered using dimensional analysis.

Part 1. Classical General Relativity

Einstein's theory of general relativity transforms Newtonian gravity into a theory that is consistent with special relativity by stating that spacetime itself is dynamic and its geometry is determined by the energy density within it. The perceived force of gravity arises from the curvature of spacetime.

This part contains a brief review of special and general relativity topics in their usual variables, then we introduce Ashtekar variables, the variables used in loop quantum gravity, using electromagnetism as a useful analogy. Finally, we include our investigations into the behavior of the Ashtekar formulation of general relativity.

1. Brief Introduction to the Mathematics of General Relativity

General relativity expands upon the formulation of special relativity in Minkowski spacetime.[2] Recall that this formulation uses four-vectors to represent quantities such as position. The position four-vector is (t, x, y, z) and distances in Minkowski spacetime are given by the spacetime interval

$$(\Delta s)^2 = -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \tag{4}$$

or

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2. \tag{5}$$

The differential form of the spacetime interval is the spacetime line element derived from the metric tensor. The metric determines how distance is measured in a spacetime. In special relativity, spacetime is approximated as flat. The flat spacetime metric $\eta_{\mu\nu}$ is defined

$$\eta_{\mu\nu} := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6}$$

An arbitrary metric is denoted $g_{\mu\nu}$, and the relationship between the spacetime line element and the metric is

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{7}$$

where $dx^\mu = (dt, dx, dy, dz)$ is a four-vector. For the metric $\eta_{\mu\nu}$, the spacetime interval is given in equation (5).

At this point, it is necessary to address why some quantities have raised indices and others have lowered indices. The placement of the indices indicates whether a quantity is covariant or contravariant. Contravariant quantities have raised indices, covariant quantities have lowered indices, and both are defined by how they transform under a change of basis. We will not rigorously define the two here, but we will outline some important properties. First, when summing over indices, one must be raised and one must be lowered. For example, it is illegal to multiply two contravariant vectors such as $x^\mu x^\mu$. Furthermore, indices are raised and lowered using the metric. To change a contravariant to a covariant vector, the transformation is

$$x_\nu = g_{\nu\mu} x^\mu. \tag{8}$$

The inverse transformation

$$x^\nu = g^{\mu\nu} x_\mu \tag{9}$$

[2]Minkowski spacetime is three-dimensional Euclidean space combined with a dimension of time.

changes a covariant vector to a contravariant vector. Note that equation (9) uses the contravariant form of the metric, called the inverse metric. It can quickly be shown from the above two equations that the metric and inverse metric obey $g_{\mu\nu}g^{\nu\alpha} = \delta^\alpha_\mu$ where δ^α_μ is the identity.

Tensors require an additional metric to raise or lower each index. For a rank-two contravariant tensor $t^{\alpha\beta}$, the covariant form is given by

$$t_{\mu\nu} = g_{\mu\alpha}g_{\beta\nu}t^{\alpha\beta}. \tag{10}$$

The transformation from a covariant tensor to a contravariant tensor is similar. It is also possible to have tensors with a mix of raised and lowered indices.

Lorentz transformations are often expressed as tensors denoted Λ^μ_ν where

$$\bar{x}^\mu = \Lambda^\mu_\nu x^\nu. \tag{11}$$

A Lorentz boost in the x-direction is

$$\Lambda^\mu_\nu = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{12}$$

where $\gamma = \frac{1}{\sqrt{1-v^2}}$ as usual. Lorentz boosts in the y- and z-directions follow the same format. The reader may easily check that for $x^\mu = (t, x, y, z)$, equation (11) recovers the usual equations for a Lorentz boost. Equation (11) may also be used for spatial transformations using the usual rotation matrices. For example, a rotation about the z-axis by angle θ is

$$\Lambda^\mu_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{13}$$

Other spatial rotations are similar. All transformations in special relativity can be expressed as a product of these transformation tensors.

As the name implies, special relativity is just a special case of general relativity, but special relativity provides a very good local approximation of general relativity because spacetime is locally Lorentzian. On larger scales, however, space can no longer be approximated as Lorentzian because energy and momentum density changes the shape of spacetime.

Mathematically describing the relationship between energy and momentum density and the geometry of spacetime is the main idea of general relativity. Before relating these two quantities, we need to represent them mathematically.

The energy and momentum density is described by the stress-energy tensor, which generalizes the stress tensor. The stress tensor T^{ij} has nine components and its indices range from 1 to 3. It is conventional that Latin indices range from 1 to 3 and indicate spatial variables whereas Greek indices range from 0 to 3 and indicate spacetime variables. The ijth component of T^{ij} gives the ith component of the force per unit area exerted on a surface orientated perpendicular to the j-direction. The stress-energy tensor $T^{\mu\nu}$ has sixteen components and includes the stress tensor. For $\mu = 1, 2, 3$ and $\nu = 1, 2, 3$, $T^{\mu\nu} = T^{ij}$. As for the other components, T^{00} is the energy density, $T^{0\nu}$ is the momentum density or energy flux for $\nu = 1, 2, 3$. The stress-energy tensor is symmetric, $T^{\mu\nu} = T^{\nu\mu}$, meaning $T^{\mu 0}$ also describes the momentum density. The stress-energy tensor encompasses all forms of energy and momentum flux and density such as mass and electromagnetic fields.

Since the geometry of spacetime is dependent upon the stress-energy tensor, it makes sense that geometry is also described by a tensor. As we have seen, it is ultimately the metric that describes the geometry, but many other theories, general relativity involves second derivatives. Hence, we need some sort of second derivative for the metric.

Consider a vector $\mathbf{V} = V^\alpha \hat{e}_\alpha$ which is expressed as a linear combination of basis vectors $\hat{e}_\alpha$. Taking the derivative of $\mathbf{V}$ with respect to a four-vector x^β, we use the product rule to find

$$\frac{\partial \mathbf{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta}\hat{e}_\alpha + V^\alpha \frac{\partial \hat{e}_\alpha}{\partial x^\beta}. \tag{14}$$

Defining Christoffel symbols $\Gamma^\mu{}_{\alpha\beta}$ as

$$\Gamma^\mu{}_{\alpha\beta}\hat{e}_\mu := \frac{\partial \hat{e}_\alpha}{\partial x^\beta}, \tag{15}$$

allows us to simplify the derivative of $\mathbf{V}$ to

$$\frac{\partial \mathbf{V}}{\partial x^\beta} = \left(\frac{\partial V^\alpha}{\partial x^\beta} + V^\mu \Gamma^\alpha{}_{\mu\beta}\right)\hat{e}_\alpha. \tag{16}$$

We see that the Christoffel symbols account for the change in the direction of the basis vectors. A simple example of a coordinate system with nonzero Christoffel symbols is polar coordinates because the direction of $\hat{r}$ and $\hat{\theta}$ change depending on their locations and hence $\hat{r}$ and $\hat{\theta}$ have nonzero derivatives. We will not do so here, but the Christoffel symbols can be calculated directly from the metric.

The Christoffel symbols are used to calculate the trajectory that an object follows in spacetime, called a geodesic. A geodesic is the shortest path that an object can take in spacetime and a generalization of a straight line in Euclidean space. More technically, a geodesic is defined using the notion of parallel-transport, which can be thought of as moving a vector an infinitesimal distance such that the resulting vector is parallel to the original vector. In non-Euclidean space, the result of the parallel-transport of a vector around a closed loop is not necessarily parallel to the original vector. A geodesic is a curve such that the tangent vector of the curve remains the tangent vector as it is parallel-transported along the curve.

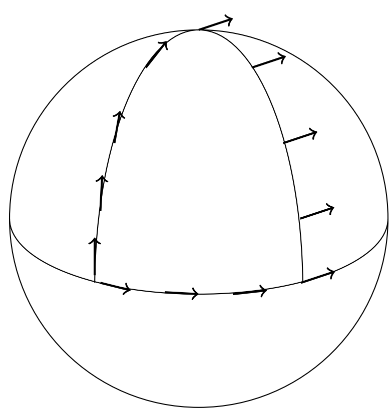

Figure 1. Parallel transport of a vector around a closed loop on the surface of a sphere

Using geodesics, we define the Riemann curvature of a manifold.[3] The Riemann curvature arises from considering the parallel-transport of a vector around a closed loop and looking at the deviation of the parallel-transported vector from the original vector. The Riemann curvature relates this deviation to the area of the

[3]A manifold is a topological space that is locally Euclidean. Any space we deal with in general relativity would be considered a manifold.

loop for each point in space. It is defined as

$$R^{\alpha}{}_{\beta\gamma\delta} = \frac{\partial \Gamma^{\alpha}{}_{\beta\delta}}{\partial x^{\gamma}} - \frac{\partial \Gamma^{\alpha}{}_{\beta\gamma}}{\partial x^{\delta}} + \Gamma^{\alpha}{}_{\gamma\epsilon}\Gamma^{\epsilon}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\delta\epsilon}\Gamma^{\epsilon}{}_{\beta\gamma}. \tag{17}$$

For a flat manifold,

$$R^{\alpha}{}_{\beta\gamma\delta} = 0, \tag{18}$$

because a flat manifold has no curvature. From the Riemann curvature, we can easily define the Ricci tensor and Ricci curvature. The Ricci tensor is defined as

$$R_{\alpha\beta} := R^{\gamma}{}_{\alpha\gamma\beta} \tag{19}$$

and the Ricci curvature is defined as

$$R := g^{\alpha\beta}R_{\alpha\beta}. \tag{20}$$

It is the Ricci tensor and the Ricci curvature which are directly related to the stress-energy tensor. The relationship is given in the famous Einstein field equations, which are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{21}$$

The first two terms on left hand side are defined as the Einstein curvature $G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$. The third term contains the cosmological constant Λ which gives the energy density of the vacuum where $T_{\mu\nu} = 0$. Physically, the cosmological constant account for the accelerating expansion of the universe.

The solutions to the Einstein field equations are rather complicated, but we will examine one, plane gravitational waves, in the next section. Other solutions require much tedious math and are not directly relevant to this thesis. Much of the information in the section was presented purely to give the required background for the Einstein field equations. However, the concepts of the metric and spacetime intervals will be important in describing plane gravitational waves and Ashtekar variables.

2. Plane Gravitational Waves

A change in energy distribution changes the geometry of spacetime, but not instantaneously. The change propagates through spacetime as a gravitational wave traveling at the speed of light. Gravitational waves are transverse waves that result from changes in energy density. A useful analogy can be drawn between gravitational waves and electromagnetic waves, which are created by accelerating charges and carry information about the electromagnetic field. Similarly, gravitational waves carry information about the metric of spacetime.

A binary black hole merger, the most well-known source of gravitational waves, was the source of the celebrated first gravitational wave detection by LIGO (Laser Interferometer Gravitational Wave Observatory) [1]. LIGO recently announced that they have also detected gravitational waves from a neutron star merger [2]. Near these collisions, the gravitational waves are quite complicated, but far away in what is known as the far field, the gravitational waves are approximately plane waves.

A plane wave is a wave which is the same in the transverse directions; its wavefronts are uniform, parallel to each other and perpendicular to the direction of travel. Electromagnetic plane waves are a familiar example of plane waves. Recall that the wave equation in vacuum is

$$\frac{\partial^2 f}{\partial z^2} - \frac{\partial^2 f}{\partial t^2} = 0. \tag{22}$$

Its most general solution is

$$f(z,t) = g(z-t) + h(z+t). \tag{23}$$

Where $g(z-t)$ represents a wave propagating in the positive z-direction and $h(z+t)$ represents a wave propagating in the negative z-direction. A gravitational plane wave uses a similar form.

In the simplest case, gravitational plane waves are derived from small perturbations in flat space. The metric is

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{24}$$

where $h_{\mu\nu}$ is the perturbation on flat spacetime. Since $\eta_{\mu\nu}$ is constant, the gravitational wave equations are given by

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right) h_{\mu\nu} = 0. \tag{25}$$

These equations are derived from the Einstein field equations in vacuum ($T_{\mu\nu} = 0$). Note that gravitational waves travel at c.

For a wave propagating in the positive z-direction, the perturbation is provided by

$$h_{\alpha\beta}(t,z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} f(t-z). \tag{26}$$

Therefore, the spacetime interval is

$$ds^2 = -dt^2 + (1 + f(t-z))dx^2 + (1 - f(t-z))dy^2 + dz^2. \tag{27}$$

Recall that electromagnetic waves have two independent polarizations. The same is true for gravitational waves. As we have defined it, $h_{\alpha\beta}(t,z)$ gives one polarization of gravitational waves, called plus-polarized. The other polarization, cross-polarized, is given by

$$h_{\alpha\beta}(t,z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} f(t-z). \tag{28}$$

By combining these two polarizations, we find $h_{\alpha\beta}$ for a general wave propagating in the z-direction:

$$h_{\alpha\beta}(t,z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & f_+(t-z) & f_\times(t-z) & 0 \\ 0 & f_\times(t-z) & -f_+(t-z) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{29}$$

In our work, we deal with polarized plane gravitational waves. Additionally, the waves we deal with propagate in only one direction because waves propagating in opposite directions could break the symmetry of the system by attracting each other and collapsing into a black hole.

3. Canonical Variables

Before introducing Ashtekar variables, the canonical variables that we will use to express general relativity, it is useful to formally define canonical variables and introduce some of their properties. From classical mechanics, the Lagrange equation is expressed in terms of the generalized coordinates q^i and $\dot{q}^i$ where i runs from 1 to n. The generalized coordinate p_i canonically conjugate to q^i is defined by

$$p_i := \frac{\partial \mathcal{L}}{\partial \dot{q}^i}. \tag{30}$$

In terms of the Hamiltonian, defined as $H = p_i \dot{q}^i - \mathcal{L}$, the canonically conjugate variables satisfy

$$\begin{aligned} \dot{q}^i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q^i}. \end{aligned} \tag{31}$$

Note that q^i and p_i are often the familiar position and momentum. Furthermore, the canonical variables provide coordinates for phase space. For a system with canonical coordinates $q^1, q^2, \ldots, q^n$, $p_1, p_2, \ldots, p_n$ the phase space is the $2n$-dimensional space labeled by these coordinates. There is a one-to-one correspondence between each point in phase space and each state of the system.

Using the canonical variables, we can define the Poisson bracket between functions of these variables. Let f and g be functions on phase space. The Poisson bracket of f and g is defined as

$$\{f, g\} := \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i}. \tag{32}$$

As can be easily verified, the Poisson brackets between canonical variables turn out to be

$$\begin{aligned} \{q^i, q^j\} &= 0 \\ \{p_i, p_j\} &= 0 \\ \{q^i, p_j\} &= \delta^i_j. \end{aligned} \tag{33}$$

These relationships are helpful in calculating the Poisson brackets of constraints, which are just functions of the canonical variables. Poisson brackets are also used to calculate how a function changes a variable. For example, the Hamiltonian evolves q_i and p_i in time:

$$\begin{aligned} \dot{q}^i &= \{q^i, H\} \\ \dot{p}_i &= \{p_i, H\}. \end{aligned} \tag{34}$$

In Section 7, the Poisson brackets are used to calculate the action of the constraints on the canonical variables. Then, in Section 9, we will see the role that Poisson brackets play an essential part in quantizing a classical theory. First, however, we will use electromagnetism as an example in order to better understand canonical variables.

4. Electromagnetism

In this section, we show that the electric field $\mathbf{E}$ is canonically conjugate to the four-potential A_μ, defined as $A_\mu = (-\phi, \mathbf{A})$. Note that we will assume that we can approximate space as flat, so our metric is $\eta_{\mu\nu}$ given by equation (6). For our purpose, it is easiest to consider the electromagnetic field tensor. Its contravariant form $F^{\mu\nu}$ is given by

$$F^{\mu\nu} = \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & B^z & -B^y \\ -E^y & -B^z & 0 & B^x \\ -E^z & B^y & -B^x & 0 \end{pmatrix}. \tag{35}$$

The covariant field tensor $F_{\mu\nu}$ is result of using the metric to lower the indices of $F^{\mu\nu}$, as shown:

$$F_{\mu\nu} = \eta_{\mu\alpha}\eta_{\beta\nu}F^{\alpha\beta}. \tag{36}$$

In lieu of explicitly calculating the Lagrangian, we will introduce it and then provide a conceptual justification. Assuming that the current density and charge density are zero, which is true in vacuum, the Lagrangian is

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{37}$$

The Lagrangian must be gauge invariant as well as invariant under spacetime transformations. The product $F_{\mu\nu}F^{\mu\nu}$ is the simplest possible quantity that can be formed from the field tensor that meets these requirements. Additionally, Maxwell's equations can be derived from the Lagrangian, giving further justification to its correctness.

Also necessary to the calculation of the conjugate coordinate to the four-potential is the expression of the Lagrangian in terms of the four-potential, which is done by expressing the electromagnetic field tensor as the asymmetric derivative of the four-potential:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{38}$$

where ∂_μ is the four-gradient defined by $\partial_\mu \to \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$ in flat space. This relationship is easily verified by direct computation.

To calculate the coordinate conjugate to A_μ, we would start with the definition

$$\pi^\alpha = \frac{\partial \mathcal{L}}{\partial \dot{A}_\alpha}. \tag{39}$$

Since $\dot{A}_\mu$ is the derivative of A_μ with respect to t, we have $\dot{A}_\mu = \partial_0 A_\mu$. Then,

$$\begin{aligned} \pi^\alpha &= \frac{\partial \mathcal{L}}{\partial\left(\partial_0 A_\alpha\right)} \\ &= \frac{\partial}{\partial(\partial_0 A_\alpha)} \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right) \\ &= \frac{1}{4} \frac{\partial}{\partial_0 A_\alpha} \left(\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\right). \end{aligned} \tag{40}$$

From here, brutal force calculations lead to the result

$$\begin{aligned} \pi^\alpha &= \partial^0 A^\alpha - \partial^\alpha A^0 \\ &= F^{0\alpha}. \end{aligned} \tag{41}$$

From equation (35), we see that $F^{00} = 0$ and $F^{0i} = E^i$ for $i = 1, 2, 3$. Thus, $\pi^\alpha = (0, \mathbf{E})$ so the electric field and the four-potential are canonical variables for electromagnetism. Note that the electric field has three components while the four-potential has four components, which indicates that there must be a constraint on the variables. In fact, there are multiple constraints: Maxwell's equations.

5. Ashtekar Variables

Like the electric field and four-potential of electromagnetism, Ashtekar variables are canonical variables for general relativity. To define Ashtekar variables, one first chooses a foliation of spacetime, which takes advantage of the $\mathcal{M} = \mathbb{R} \times M$ structure of spacetime where $\mathcal{M}$ is associated with spacetime, $\mathbb{R}$ is associated with time, and M is associated with space. Choosing a foliation means choosing a particular time and space. Then, at each point on the spatial manifold M, there is an spatial metric q_{ab}. The inverse triad e_i^a is related to this metric through

$$e_i^a e_j^b q_{ab} = \eta_{ij}. \tag{42}$$

Through this equation, the inverse triad relates the spatial metric to the flat space metric η_{ij}. The a or b index correspond to each spatial direction on M. The i or j index of the inverse triad corresponds to the expression of the inverse triad components in an orthonormal basis of the tangent space at a point on M.

These components can take values from the Lie algebra $su(2)$ because of the isomorphism between $so(3)$ and $su(2)$.

The inverse densitized triad E_i^a is obtained by rescaling the inverse triad using the metric:

$$E_i^a = \sqrt{\det(q)}e_i^a. \tag{43}$$

The inverse densitized triad is one of the Ashtekar variables. The conjugate variable to the inverse densitized triad is the connection A_i^a. The connection is defined in terms of the curvature K_a^i of the spatial manifold and the connection associated with the inverse triad given by

$$\Gamma_a^i := e_a^i \Gamma_{bc}^a. \tag{44}$$

Then the connection is defined as

$$A_a^i := \Gamma_a^i + K_a^i. \tag{45}$$

The connection is used in the parallel transport of vectors between points in space and so allows for the comparison of vectors from different locations.

These variables are similar to the electric field $\mathbf{E}$ and the four-potential A_μ of electromagnetism. Classical general relativity can be completely described by Ashtekar variables provided that they satisfy three constraints: the Gauss constraint, the diffeomorphism constraint, and the Hamiltonian constraint. The constraints ensure the the Ashtekar formulation has to necessary symmetries of general relativity. Here, we give the form the constraints as they are given by Giesel [11].

The Gauss constraint is the akin to Gauss's law in vacuum in electromagnetism. It accounts for rotational invariance and has the form

$$\mathcal{D}_a E_j^a = 0, \tag{46}$$

which is analogous to $\nabla \cdot \mathbf{E} = 0$. It then turns out that $\mathcal{D}_a$ is defined through

$$\mathcal{D}_a E_j^a := \partial_a E_j^a + \epsilon_{jk}{}^{\ell} A_a^k E_\ell^a = G_j \tag{47}$$

where $\epsilon_{jk}{}^{\ell}$ is the Levi-Cevita symbol.

The diffeomorphism constraint D_a ensures the system behaves the same regardless of the spatial coordinate system. It generates a diffeomorphism on M given by

$$D[N] = \int_M N^a D_a dx^3 \tag{48}$$

for a spatial shift N^a. Assuming that the Gauss constraint is satisfied, the diffeomorphism constraint is

$$D_a = F_{ab}^j E_j^b \tag{49}$$

where F_{ab}^j is the curvature associated with the connection given by

$$F_{ab}^j = \partial_a A_b^j - \partial_b A_a^j + \epsilon^j{}_{k\ell} A_a^k A_b^\ell. \tag{50}$$

The curvature F_{ab}^j is analogous to the electromagnetic field tensor $F_{\mu\nu}$. The major difference comes in the last term of equation (50), which accounts for the group structure of the variables.

The Hamiltonian constraint H ensures the system behaves the same regardless of the spacetime coordinate system. It is given by

$$H = 4\pi G \gamma^2 \epsilon_j{}^{mn} E_m^a E_n^b \left(F_{ab}^j - (1+\gamma^2)\epsilon^{jk\ell} K_a^k K_b^m \right) \tag{51}$$

where G is Newton's gravitational constant and γ is the Barbero-Immirzi parameter.[4]

[4]The Barbero-Immirzi parameter is currently undetermined but could potentially be calculated from physical situations. One such calculation is the computation of black hole entropy. At this point, we can only say $\gamma \in \mathbb{C}$ and note that each choice of γ gives a different set of canonical variables. However, the mathematics are only fully worked out for the case $\gamma \in \mathbb{R}$.

In addition to the aforementioned constraints, plane gravitational waves must also satisfy the unidirectional constraint which ensures the waves propagate in only one direction. There are two unidirectional constraints: one for right-moving waves and another for left-moving waves.

6. Constraints in One Dimension

Since this thesis deals exclusively with plane gravitational waves, it is useful to make a number of simplifications to the constraints, and then present a simplified unidirectional constraint. We set our coordinate system so that the waves travel in the z-direction. Then the x and y symmetry of the waves allows us to significantly reduce the amount of variables involved. We do this simplification in the polarized Gowdy model (see [7]) and the details are beyond the scope of this thesis. For our purposes, it is important to note that after the simplification, the only remaining variables are the spatial components of the triad E^x, E^y, and $\mathcal{E}$, and the spatial components of the connection K_x, K_y, and $\mathcal{A}$. The variables that have been renamed differ from earlier versions by a constant. For the forms of the constraints given below, the Gauss constraint has been solved and used to simplify the other constraints.

With these simplifications made, we give the simplified constraints in the same forms as are used by Hinterleitner and Major [13]. The right-moving unidirectional constraint is

$$U_+ = E^x K_x + E^y K_y + \mathcal{E}' \tag{52}$$

where the prime denotes the derivative with respect to z. We exclusively work with the right-moving constraint in this thesis, but we will also give the left-moving constraint for the sake of completeness. It is

$$U_- = E^x K_x + E^y K_y - \mathcal{E}'. \tag{53}$$

The one-dimensional diffeomorphism constraint is

$$D = E^x K_x' + E^y K_y' + \mathcal{E}'\mathcal{A}. \tag{54}$$

The one-dimensional Hamiltonian constraint is

$$H = \frac{1}{\mathcal{E}} E^x K_x E^y K_y + (E^x K_x + E^y K_y)\mathcal{A} - \frac{1}{4}\frac{\mathcal{E}'^2}{\mathcal{E}} - \mathcal{E}'' - \frac{1}{4}\mathcal{E}\left[\left(\ln\left(\frac{E^y}{E^x}\right)\right)'\right]^2 + \frac{1}{2}\mathcal{E}'\left(\ln\left(E^x E^y\right)\right)'. \tag{55}$$

We used these forms of the constraints to verify the algebra of the classical theory, which is defined by the actions of the constraints with test functions $f(z)$ and $g(z)$. The test functions control the extent of the transformation generated by the constraint. For example, the change to E^x over time is $\{E^x, H[f]\}$ where f controls how much time passes. Similarly, the change to E^x by a diffeomorphism is $\{E^x, D[f]\}$ where in this case f controls the size of the diffeomorphism. The Poisson brackets between constraints give the difference in the difference in the result when the order of the action by two constraints is switched. Setting aside the Gauss constraint, we have the following Poisson brackets:

$$\{H[f], H[g]\} = D[f'g - fg'] \tag{56}$$

$$\{D[f], H[g]\} = H[f'g - fg'] \tag{57}$$

$$\{U_+[f], H[g]\} = U_+[-fg'] \tag{58}$$

$$\{U_+[f], D[g]\} = U_+[f'g]. \tag{59}$$

From these Poisson brackets, we see that the order that the constraints act a point matters.

7. Spacetime Diagrams

To better understand the actions of the constraints and the relationships between them, we will look at their actions represented on a spacetime diagram, but first, we need to define active and passive transformations. The important distinction between the two is that active transformations move the location of a point whereas passive transformations change the coordinate system of a point. A passive transformation given by the inverse of an active transformation will give the same final point as that active transformation. Similarly, an active transformation given by the inverse of a passive transformation will the give same result as the passive transformation.

For example, consider a point (x_0, y_0) located in the xy-plane. Moving the point over by (a, b) will result in the new point $(x_0 + a, y_0 + b)$ as shown in Figure 2. The passive transformation given by the inverse of this active transformation moves the coordinate system by $(-a, -b)$ as shown in Figure 3. Expressed in terms of the new coordinate system, original point becomes $(x_0 + a, y_0 + b)$, just as it would under the active transformation. We will use the relationship between active and passive transformations to express the constraints as active transformations of points on a spacetime diagram.

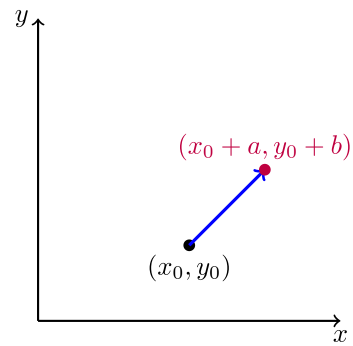

Figure 2. Active transformation of the point (x_0, y_0) by (a, b)

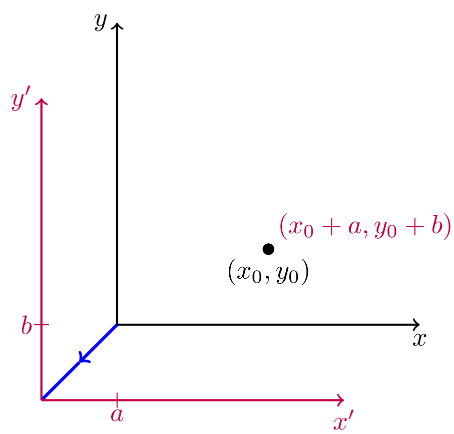

Figure 3. Passive transformation of the point (x_0, y_0) by $(-a, -b)$

The diffeomorphism, Hamiltonian, and unidirectional constraints can all be represented as infinitisimal transformations on a spacetime diagram. Because we are dealing with plane waves, E^a_i and A^i_a do not depend on x and y, and therefore, we are only concerned with transformations of z and t.

The diffeomorphism constraint represents a spatial transformation. To find its action on a graph, we need to calculate how it changes the six canonical variables by calculating the Poisson bracket of each variable with $D[f]$. As an example, we will calculate $\{E^x, D[f]\}$. The calculations for K_x, E^y, K_y, $\mathcal{E}$, and $\mathcal{A}$ are very similar.

Since E^x and f both depend on where in space they are evaluated, we need to specify at which point each one is evaluated. Choosing to evaluate E^x at $\tilde{z}$ and f at z, the Poisson bracket is

$$\begin{aligned} \{E^x(\tilde{z}), D[f(z)]\} &= \left\{E^x(\tilde{z}), \int f(z)D(z)\, dz\right\} \\ &= \left\{E^x(\tilde{z}), \int f(z)\left(E^x(z)K_x'(z) + E^y(z)K_y'(z) + \mathcal{E}'(z)\mathcal{A}(z)\right)\, dz\right\} \end{aligned} \tag{60}$$

The integral accounts for the fact that we are checking the whole space for the points where $\tilde{z}$ and z are the same and runs from negative infinity to infinity. Next, since the Poisson bracket is bilinear, we find

$$\begin{aligned} \{E^x(\tilde{z}), D[f(z)]\} = \left\{E^x(\tilde{z}), \int f(z)E^x(z)K_x'(z)\, dz\right\} + \left\{E^x(\tilde{z}), \int f(z)E^y(z)K_y'(z)\, dz\right\} \\ + \left\{E^x(\tilde{z}), \int f(z)\mathcal{E}'(z)\mathcal{A}(z)\, dz\right\} \end{aligned} \tag{61}$$

Recall from equation (33) in Section 3 that the Poisson bracket between two canonical variables is zero unless it is between the same components of conjugate variables. Therefore, the Poisson brackets of E^x with E^x, E^y, K_y, $\mathcal{E}$, and $\mathcal{A}$ vanish. Only $\{E^x, K_x\}$ is nonzero. Since only the first term in equation (61) contains K_x in the second slot, the other terms vanish, giving

$$\{E^x(\tilde{z}), D[f(z)]\} = \left\{E^x(\tilde{z}), \int f(z)E^x(z)K_x'(z)\, dz\right\}. \tag{62}$$

The next step involves integration by parts. In order for the Poisson bracket to converge, f must go to zero at infinity and negative infinity. Hence, we have

$$\begin{aligned} \int_{-\infty}^{\infty} f(z)E^x(z)K_x'(z)\, dz &= f(z)E^x(z)K_x(z)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \left(f(z)E^x(z)\right)' K_x(z)\, dz \\ &= -\int_{-\infty}^{\infty} \left(f(z)E^x(z)\right)' K_x(z)\, dz \end{aligned} \tag{63}$$

By using the above equation and a version of the product rule, we find

$$\begin{aligned} \{E^x(\tilde{z}), D[f(z)]\} &= \left\{E^x(\tilde{z}), -\int \left(f(z)E^x(z)\right)' K_x(z)\, dz\right\} \\ &= -\int dz \left(f(z)E^x(z)\right)' \{E^x(\tilde{z}), K_x(z)\} - \int dz K_x(z) \left\{E^x(\tilde{z}), \left(f(z)E^x(z)\right)'\right\} \\ &= -\int dz \left(f(z)E^x(z)\right)' \{E^x(\tilde{z}), K_x(z)\}. \end{aligned} \tag{64}$$

The generalization of equation (33) to a function space is $\{E^x(\tilde{z}), K_x(z)\} = \delta(\tilde{z} - z)$ where $\delta(\tilde{z} - z)$ is the Dirac delta function. Therefore, the Poisson bracket evaluates to

$$\begin{aligned} \{E^x(\tilde{z}), D[f(z)]\} &= -\int dz \left(f(z)E^x(z)\right)' \delta(\tilde{z} - z) \\ &= -\left(f(\tilde{z})E^x(\tilde{z})\right)'. \end{aligned} \tag{65}$$

This result shows that a given point $\tilde{z}$ with triad $E^x(\tilde{z})$ will have triad $E^x(\tilde{z}) - (f(\tilde{z})E^x(\tilde{z}))'$ after the diffeomorphism. For a constant shift, this triad can be written as

$$\begin{aligned} E^x(\tilde{z}) - (f(\tilde{z})E^x(\tilde{z}))' &= E^x(\tilde{z}) - f(\tilde{z})E^{x\prime}(\tilde{z}) \\ &\approx E^x(\tilde{z} - f(\tilde{z})) \end{aligned} \tag{66}$$

by using a Taylor series approximation about $\tilde{z}$.

Suppose f is a shift by δ, so $f(z) = \delta$ for all z. Then for any point z, the triad before the diffeomorphism is $E^x(z)$ and the triad after the diffeomorphism is $E^x(z-\delta)$, so the diffeomorphism essentially takes the triad $E^x(z-\delta)$ located at $z-\delta$ and moves it over by δ to put it at z. By additional brute force calculations, one can verify that the diffeomorphism has the same action on the other five canonical variables. Expressed another way, the diffeomorphism constraint is a spatial translation that takes the variables on a spacetime diagram at (t, z) and moves them to $(t, z+\delta)$, which is given by

$$D[\delta] : (t, z) \to (t, z + \delta). \tag{67}$$

The other constraints also generate changes on a spacetime diagram that can be verified using Poisson brackets. The unidirectional constraint represents a shift in the direction of a light ray given by

$$U_+[\beta] : (t, z) \to (t + \beta, z + \beta). \tag{68}$$

The Hamiltonian constraint for plane waves represents a spacetime transformation, which is a Lorentz boost. A Lorentz boost can be written in terms of the rapidity α where $\tanh\alpha = v$. Usually a Lorentz boost is given as passive transformation, which is a change of reference frame:

$$\begin{aligned} z' &= z\cosh\alpha - t\sinh\alpha \approx z - \alpha t \\ t' &= t\cosh\alpha - z\sinh\alpha \approx t - \alpha z, \end{aligned} \tag{69}$$

where small angle approximations are used because we are dealing with infinitesimal boosts. We are representing the constraints as active transformations, which move a point within a single reference frame. Recall that an active transformation is the inverse transformation of the equivalent passive transformation. Written as an active transformation, an infinitesimal Lorentz boost is

$$\begin{aligned} z &\to z + \alpha t \\ t &\to t + \alpha z. \end{aligned} \tag{70}$$

Therefore, the Hamiltonian constraint is

$$H[\alpha z] : (t, z) \to (t + \alpha z, z + \alpha t). \tag{71}$$

The test function for the Hamiltonian constraint is αz because it needs to be a function of z rather than t. Also, the initial points are placed on the z-axis where $t = 0$.

With these test functions defined, the constraint algebra from equations (57) and (58) becomes

$$\{D[\delta], H[\alpha z]\} = H[-\alpha\delta] \tag{72}$$

$$\{U_+[\beta], H[\alpha z]\} = U_+[-\alpha\beta], \tag{73}$$

showing that the order that constraints act on a point does matter. The first relation states that performing a diffeomorphism then a Lorentz boost on an initial point will not result in the same final point as performing a Lorentz boost then a diffeomorphism on that same initial point. The two final points are separated by a Lorentz boost $H[-\alpha\delta]$, as shown in Figure 4. Similarly, the second relation is shown in Figure 5.

The algebra of Poisson brackets is significant because it carries over into quantum mechanics as the algebra of the commutators. A strong understanding of the classical algebra will provide insight into the quantization process.

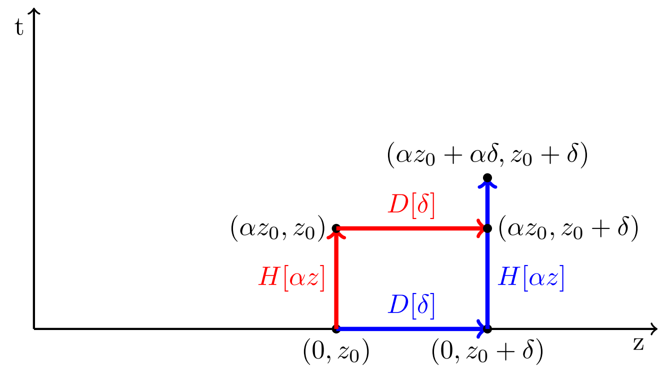

Figure 4. This spacetime diagram shows the algebra of a spatial translation and a Lorentz boost. The blue vectors show a diffeomorphism followed by a Lorentz boost. The red vector show a Lorentz boost followed by a diffeomorphism from the same initial point.

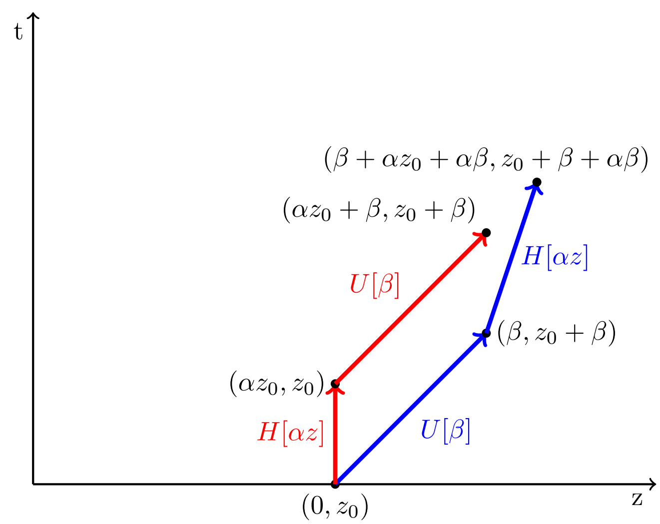

Figure 5. This spacetime diagram shows the algebra of a Lorentz boost and propagation along a light cone. The blue vectors show propagation along a light cone followed by a Lorentz boost. The red vectors show a Lorentz boost followed by propagation along a light cone.

Part 2. Loop Quantum Gravity

Quantum gravity aims to answer one of the biggest questions of theoretical physics: how does one build a theory encompassing both general relativity and quantum mechanics? Loop quantum gravity is a theory of quantum gravity which uses canonical quantization to quantize space as described classical general relativity. The resulting theory is expressed using a graph called a spin network. This part first introduces holonomies and explains their significance to quantum mechanics. Then, we explain the process of canonical quantization and apply it in loop quantum gravity, including a description of spin networks.

8. Holonomies

The motivation for using a spin network comes partly from the need for holonomies, which are used to express quantities that depend on gauge in a gauge invariant form. Once again, we will use electromagnetism to illustrate to necessity of holonomies. Classically, we are used to working with the electric field and the magnetic field, gauge invariant quantities. However, we just showed that the electric field and the four-potential are canonical variables for electromagnetism, so it should be possible to express the evolution of the system in terms of these variables. As it happens, it is sometimes necessary to consider $\mathbf{E}$ and A_μ as the variables of electromagnetism. This need arises in the description of the quantum phenomenon called the Aharonov-Bohm effect.

The Aharonov-Bohn effect, which has been observed experimentally [21], predicts that a beam of charged particles split by a solenoid will exhibit an interference pattern that is dependent upon the vector potential despite the fact that the beam travels through a region where the magnetic field is zero, as shown in Figure 6.

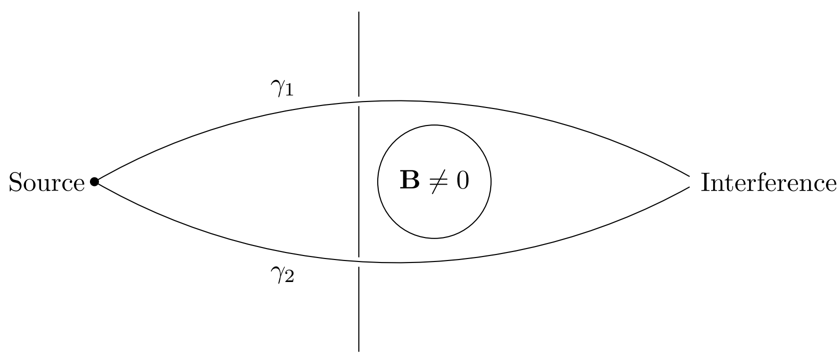

Figure 6. The Aharonov-Bohm effect

A full explanation requires the path integral formulation of quantum mechanics. Most importantly, the time evolution of a particle is given by

$$|\psi(t')\rangle = e^{i\mathcal{S}/\hbar} |\psi(t_0)\rangle \tag{74}$$

where $\mathcal{S}$ is the familiar action from classical mechanics,

$$\mathcal{S} = \int_{t_0}^{t'} \mathcal{L}\, dt. \tag{75}$$

For a charged particle moving in a vector potential, the total phase $\frac{\mathcal{S}}{\hbar}$ can be broken into two parts: a phase ϕ arising from the vector potential and a phase ϕ_0 arising from the remaining part of the Lagrangian. We can ignore ϕ_0 because it is the same for both beams of particles. To calculate ϕ, we need to know that the

Lagrangian term arising from the vector potential is $q\mathbf{A} \cdot \mathbf{v}$ for a particle with charge q moving with velocity $\mathbf{v}$.[5] Therefore, the phase acquired from the vector potential is

$$\phi = \frac{1}{\hbar} \int_{t_0}^{t'} q\mathbf{A} \cdot \mathbf{v} dt = \frac{q}{\hbar} \int_C \mathbf{A} \cdot d\mathbf{s} \tag{76}$$

showing that the phase depends only on the line integral of the vector potential along the trajectory C of the particle.

To calculate the phase of the particles in the experiment shown in Figure 6, let γ_1 and γ_2 be the two paths that the charged particles take. Note that the solenoid is between the two paths and that the magnetic field is nonzero only within the solenoid. However, the vector potential in nonzero along the paths γ_1 and γ_2. The phase change due to the vector potential of a particle traveling along γ_1 is given by

$$\phi_1 = \frac{q}{\hbar} \int_{\gamma_1} \mathbf{A} \cdot d\mathbf{s}. \tag{77}$$

Similarly, the phase change of a particle traveling along γ_2 is given by

$$\phi_2 = \frac{q}{\hbar} \int_{\gamma_2} \mathbf{A} \cdot d\mathbf{s}. \tag{78}$$

From equations (77) and (78), we find the phase difference between the beams of particles when the beams interfere. The phase difference is

$$\begin{aligned} \Delta\phi &= \phi_1 - \phi_2 \\ &= \frac{q}{\hbar} \int_{\gamma_1} \mathbf{A} \cdot d\mathbf{s} - \frac{q}{\hbar} \int_{\gamma_2} \mathbf{A} \cdot d\mathbf{s} \\ &= \frac{q}{\hbar} \oint_{\gamma} \mathbf{A} \cdot d\mathbf{s}. \end{aligned} \tag{79}$$

Using Stokes' theorem, we find

$$\begin{aligned} \Delta\phi &= \frac{q}{\hbar} \int_S \vec{\nabla} \times \mathbf{A} \cdot d\mathbf{a} \\ &= \frac{q}{\hbar} \int_S \mathbf{B} \cdot d\mathbf{a} \\ &= \frac{q\Phi}{\hbar} \end{aligned} \tag{80}$$

where S is the surface bounded by γ and Φ is the magnetic flux through S.

At first, this result seems incredibly problematic because it appears that the vector potential itself has direct observable consequences. The role of the vector potential extends beyond a mathematical tool to calculate the magnetic field, forcing us to accept its physical importance. However, the vector potential is not gauge invariant, but anything we observe cannot depend on a choice of gauge. Fortunately, we can define a holonomy to solve the problem. A holonomy along γ is

$$h_\gamma := e^{i \frac{q}{\hbar} \int_\gamma \mathbf{A} \cdot d\mathbf{s}}. \tag{81}$$

The above holonomy is always gauge invariant along a closed path.

Holonomies are important in loop quantum gravity because they are used to create a quantity with physical significance from a gauge invariant quantity. Since the connection A_a^i is not gauge invariant, we

[5] The proof of this fact requires calculating the equations of motion from the Lagrangian

$$\mathcal{L} = \frac{1}{2} m\mathbf{v}^2 - q\varphi + q\mathbf{A} \cdot \mathbf{v}$$

and noting that it yields the Lorentz force.

take the holonomy of A_a^i to have physical significance. As we will see in Section 10, the holonomy used in loop quantum gravity is slightly more complicated. It is a matrix whose trace is gauge invariant provided the holonomy is along a closed loop. The necessity of closed loops for holonomies gives spin networks their characteristic loop structure. To reach holonomies on spin networks however, we first must introduce and use canonical quantization.

9. Canonical Quantization

Canonical quantization is the process used in loop quantum gravity. It begins with a classical theory expressed in terms of canonically conjugate variables. The variables and constraints become quantum operators, and the Poisson brackets between quantities become commutators between operators. Symbolically, for two classical quantities A and B, we have the relations

$$\begin{aligned} A &\rightarrow \hat{A} \\ B &\rightarrow \hat{B} \\ \{A, B\} &\rightarrow [\hat{A}, \hat{B}] = i\hbar\widehat{\{A, B\}}. \end{aligned} \tag{82}$$

It follows that for canonically conjugate variables x^i and p_i, the commutator is $[\hat{x}^i, \hat{p}_j] = i\hbar\delta^i_j$. Furthermore, we can express $\hat{x}^i$ and $\hat{p}_i$ in the $|x^i\rangle$ basis as

$$\hat{x}^i |x^i\rangle = x^i |x^i\rangle \tag{83}$$

$$\hat{p}_i |x^i\rangle = -i\hbar\frac{\partial}{\partial x^i} |x^i\rangle \tag{84}$$

We will justify these choices by examining the Hamiltonian operator of single particle mechanics and showing that it yields the familiar time independent Schrödinger equation if we make the choices given in equations (83) and (84). Classically, the Hamiltonian is equal to the sum of the kinetic and potential energies. The same holds true in quantum mechanics but the Hamiltonian becomes a quantum operator. We find

$$\begin{aligned} \langle x^i| \hat{H} |\psi\rangle &= \langle x^i| \left(\hat{T} + \hat{U}\right) |\psi\rangle \\ &= \langle x^i| \left(\frac{\hat{p}_i^2}{2m} + U(\hat{x}^i)\right) |\psi\rangle . \end{aligned} \tag{85}$$

Expressing $\hat{x}^i$ and $\hat{p}_i$ in the $|x^i\rangle$ basis as in equations (83) and (84) gives

$$\begin{aligned} \langle x^i| \hat{H} |\psi\rangle &= \langle x^i| \left(\frac{(-i\hbar)^2}{2m}\left(\frac{\partial}{\partial x^i}\right)^2 + U(x^i)\right) |\psi\rangle \\ &= \langle x^i| \left(\frac{-\hbar^2}{2m}\nabla^2 + U(x^i)\right) |\psi\rangle . \end{aligned} \tag{86}$$

Since the Hamiltonian is the total energy, we expect the eigenvalues of the Hamiltonian to be the energy eigenvalues, meaning $\hat{H}|\psi\rangle = E|\psi\rangle$. Thus, we have just found the time independent Schrödinger equation for an energy eigenstate:

$$\langle x^i| \left(\frac{-\hbar^2}{2m}\nabla^2 + U(x^i)\right) |\psi\rangle = \langle x^i| E |\psi\rangle . \tag{87}$$

Although this "derivation" was only done for a certain case, equations (83) and (84) hold more generally for canonical variables and we will use a very similar form in our quantization.

Also recall that the energy eigenstates form a basis for all possible states of a wavefunction meaning any arbitrary state can be written as a linear combination of energy eigenstates:

$$|\psi\rangle = \sum_{n=1}^{\infty} c_n |\psi_n\rangle \tag{88}$$

where $|\psi_n\rangle$ are energy eigenstates and c_n are complex coefficients satisfying $\sum_{n=1}^{\infty} |c_n|^2 = 1$. The need to find a basis for all possible states carries over into loop quantum gravity, as is discussed in the next section.

10. Spin Networks and Quantum States

A spin network forms a basis for the states of spatial geometry [6]. As described by Major [17], a spin network is a three-dimensional labeled graph[6] where all edges and some vertices[7] have a label associated with them. The labels denote the spin states. The name spin network comes from the fact that when edges meet at a vertex, their labels obey the same rules that quantum spin numbers obey. Figure 7 shows an example spin network.

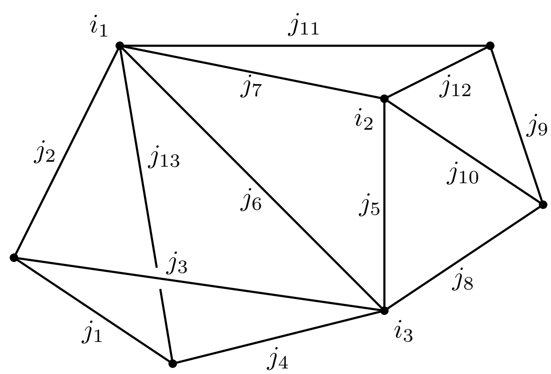

Figure 7. A spin network where edges are labeled by spins j_n and vertices are labeled by intertwiners i_m

To find information about geometry such as areas [4], [20], angles [18], and volumes [5], [14], [15], [16], [20] from a spin network, one uses quantum operators whose eigenvalues are in expressed in terms of the labels. The volume operator gives the volume of a region of space containing a particular vertex, depending on where the volume operator acts. The resulting volume depends on the intertwiner of the vertex. Similarly, the area operator gives the area of a surface pierced by a particular edge and the result depends on the label of the edge. The angle operator acts on two edges meeting at a vertex. In this way, a spin network provides a discrete description of geometry.

The motivation for spin networks comes from the need for a discrete analog to continuous space formulated on a manifold. Continuous space leads to many difficulties in the form of divergent quantities in both general relativity and quantum mechanics. For example, at the center of a black hole, the curvature of spacetime diverges to infinity. Similarly, many quantities calculated in quantum field theory must be renormalized to eliminate divergences. Since continuity enables divergences, discretizing space will eliminate this problem.

Spin networks are also a natural representation of holonomies. As discussed in Section 8, the connection A_a^i is not gauge invariant, but the trace of its holonomy around a closed loop of a spin network is gauge

[6]A graph is defined as a set of vertices and a set of edges, expressed as $G = \{\mathbf{v}, \mathbf{e}\}$.

[7]Vertices are only labeled if they are the intersection of more than three edges. The label, called an intertwiner, describes how to express the vertex in terms of trivalent vertices (intersections of three edges).

invariant. Taking the trace of the holonomy is necessary because A^i_a is a matrix so the holonomy is also a matrix. The holonomy of A along a path γ is defined as

$$h_\gamma(A) = \mathrm{P}\exp\left[\int_\gamma A^i_a \tau_i \dot{\gamma}^a dt\right] = \mathrm{P}\exp\left[\int_\gamma A^i_a \tau_i dx^a\right] \tag{89}$$

where τ_i are defined in terms of the Pauli matrices[8] and the exponential is a path-ordered exponential, which comes from taking the ordered product along segments of a line as the length of each segment approaches zero. Mathematically, it is defined as

$$\mathrm{P}\exp\left[\int_\gamma A^i_a \tau_i dx^a\right] = \lim_{n\to\infty} \prod_{k=1}^{n} \left(1 + A^i_a(x_k)\tau_i dx^a_k\right). \tag{90}$$

Despite the more complicated formulation, these holonomies are still similar to holonomies in electromagnetism. Just as there is a holonomy of the vector potential along every path in space, there is a holonomy of the connection associated with every edge of a spin network.

For the case of plane gravitational waves, we can simplify the situation considerably by looking at holonomies on a one-dimensional graph rather than on a three-dimensional graph. The symmetry of a plane gravitational wave allows us to represent its spin network as a line with nodes placed along it, as shown in Figure 8. The wave propagates in the direction of the edges e_i, and information about variables in that direction are given by integer labels k_{e_i} along the edges. Labels μ_{v_i} and ν_{v_i} at the vertices v_i contain information about variables in the two directions perpendicular to the direction of propagation.

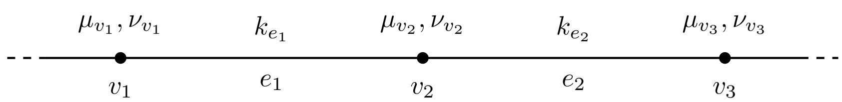

FIGURE 8. Spin network representing plane gravitational wave where each edge e_i has an integer label k_{e_i} and each vertex v_i has integer labels μ_{v_i} and ν_{v_i}

The quantum states are represented by products of holonomies along edges and at vertices. The equations in the rest of this section are taken from Banerjee and Date [8]. A holonomy of the connection along an edge is

$$h_e[\mathcal{A}] := e^{i\frac{k_e}{2}\int_e \mathcal{A}\,dz} \tag{91}$$

where k_e is the integer label assign to edge e. At a vertex, the holonomies are point holonomies due to the reduction from three to one dimensions. The point holonomies are

$$h_v[K_x] := e^{i\frac{\mu_v}{2}K_x} \tag{92}$$

$$h_v[K_y] := e^{i\frac{\nu_v}{2}K_y} \tag{93}$$

where μ_v and ν_v are the integer labels assigned to vertex v. Note that these holonomies are no longer matrices because the matrices τ_i can be replaced by the imaginary unit i with no effects on the theory. The lack of effect of Pauli matrices on our results is covered in the Appendix.

Using the holonomies, we then represent a basis state of the graph by

$$\langle K_x, K_y, \mathcal{A}|\vec{\mu}, \vec{\nu}, \vec{k}\rangle = \prod_{e\in G} e^{i\frac{k_e}{2}\int_e \mathcal{A}\,dz} \prod_{v\in G} e^{i\frac{\mu_v}{2}K_x} e^{i\frac{\nu_v}{2}K_y}. \tag{94}$$

[8]More specifically, $\tau_i = -\frac{i}{2}\sigma_i$ where σ_i are the Pauli matrices.

Because the triad and connection are canonically conjugate variables, we can express them as quantum operators. In the basis of the connection variables, we have

$$\hat{E}^x = -i\gamma\ell_p^2 \frac{\delta}{\delta K_x} \tag{95}$$

$$\hat{E}^y = -i\gamma\ell_p^2 \frac{\delta}{\delta K_y} \tag{96}$$

$$\hat{\mathcal{E}} = -i\gamma\ell_p^2 \frac{\delta}{\delta \mathcal{A}}. \tag{97}$$

To find $\hat{E}^x$ and $\hat{E}^y$, we integrate along an edge I, summing over the vertices along that edge. For the eigenvalues, we find

$$\int_I \hat{E}^x \left|\vec{\mu}, \vec{\nu}, \vec{k}\right\rangle = \frac{\gamma\ell_p^2}{2} \sum_{v\in I} \mu_v \left|\vec{\mu}, \vec{\nu}, \vec{k}\right\rangle \tag{98}$$

$$\int_I \hat{E}^y \left|\vec{\mu}, \vec{\nu}, \vec{k}\right\rangle = \frac{\gamma\ell_p^2}{2} \sum_{v\in I} \nu_v \left|\vec{\mu}, \vec{\nu}, \vec{k}\right\rangle . \tag{99}$$

For $\hat{\mathcal{E}}$, the eigenvalues only depend of k at each end of the interval, denoted k_{e^+} and k_{e^-}. So we find

$$\hat{\mathcal{E}} \left|\vec{\mu}, \vec{\nu}, \vec{k}\right\rangle = \frac{\gamma\ell_p^2}{2} \frac{k_{e^+} + k_{e^-}}{2} \left|\vec{\mu}, \vec{\nu}, \vec{k}\right\rangle . \tag{100}$$

11. Volume Operator

As an example of a quantum operator, we will briefly discuss the volume operator. Classically, the volume is given by

$$V = \int \sqrt{g}\; d^3x \tag{101}$$

where g is the metric, so $\sqrt{g} = \sqrt{|\mathcal{E}|E^x E^y}$. Banarjee and Date provide a way to quantize the volume [8]. First, the integral is broken up into the sum of n integrals, each over a length ε. We find

$$V \approx \sum_{i=1}^{n} \sqrt{|\mathcal{E}|} \sqrt{\left|\int_{z_i}^{z_i+\varepsilon} E^x\right|} \sqrt{\left|\int_{z_i}^{z_i+\varepsilon} E^y\right|}. \tag{102}$$

By changing the variables to operators, we find the volume operator:

$$\hat{V} := \sum_{i=1}^{n} \sqrt{|\hat{\mathcal{E}}|} \sqrt{\left|\int_{z_i}^{z_i+\varepsilon} \hat{E}^x\right|} \sqrt{\left|\int_{z_i}^{z_i+\varepsilon} \hat{E}^y\right|}. \tag{103}$$

Using equations (95) to (97), the eigenvalues of the volume operator are

$$\hat{V} \left|\mu_v, \nu_v, k_e\right\rangle = \frac{1}{\sqrt{2}} \left(\frac{\gamma\ell_p^2}{2}\right)^{3/2} \sum_{v\in I} \sqrt{|\mu_v||\nu_v||k_{e^+} + k_{e^-}|}\; \left|\mu_v, \nu_v, k_e\right\rangle . \tag{104}$$

Note that if $\mu_v = 0$, $\nu_v = 0$, or $k_{e^+} + k_{e^-} = 0$, then the vertex has no volume. We refer to a vertex with no volume as degenerate space.

Part 3. Quantization of the Unidirectional Constraint

In this section, we complete part of the canonical quantization process by creating a unidirectional operator from the unidirectional constraint. We act on a node of an arbitrary one-dimensional spin network with the unidirectional operator and find quantum states that satisfy the operator. We then check that the quantum states are normalizable.

We create multiple quantizations of the unidirectional constraint to account for the various choices we have to make in the quantization process. The main choices are which of two possible ways to approximate $\hat{K}_x$ and $\hat{K}_y$ using Taylor expansions and how to order the terms in U_+. We will address these items in more depth individually.

12. Quantization Process

The quantization process follows the steps laid out in Section 9. The canonically conjugate variables used for quantization are the Ashtekar variables with classical general relativity expressed as the Gauss, diffeomorphism, and Hamiltonian constraints. Plane gravitational waves are expressed through the unidirectional constraint. These constraints are written in terms of the Ashtekar variables. The first step in quantization is to promote each of the Ashtekar variables to a quantum operator and then promote each of the constraints to a quantum operator. This project focuses on promoting the unidirectional constraint to the unidirectional operator, but in order to complete the quantization, all constraints must be promoted to operators. To quantize the unidirectional constraint, each term is sent at an operator, as shown:

$$U_+[f] = \int_I dz f(E^x K_x + E^y K_y + \mathcal{E}') \to \hat{U}_+ = \sum_{v \in I} f(\widehat{E^x K_x} + \widehat{E^y K_y} + \hat{\mathcal{E}}'). \tag{105}$$

Note that the terms $\widehat{E^x K_x}$ could be ordered as either $\hat{E}^x \hat{K}_x$ or $\hat{K}_x \hat{E}^x$ with similar orderings for $\widehat{E^y K_y}$. We quantize constraint both ways because there is no clear reason for choosing one ordering over the other.

After creating the unidirectional operator, we find the states that satisfy it, which are the states such that

$$\hat{U}_+ |\psi\rangle = 0 \tag{106}$$

is true at all nodes. In this work, we investigate states at a single node, which could be combined to give states at all nodes. Recall that $|\mu\ \nu\rangle$ a basis state at a single node. By letting $m = \frac{\mu}{\mu_0}$ and $n = \frac{\nu}{\nu_0}$ for some constants μ_0 and ν_0, we can represent $|\mu\ \nu\rangle$ as $|m\ n\rangle$. Then an arbitrary state at a node is a superposition of basis states with amplitudes a_{mn} denoted as $|\psi\rangle = \sum_{m,n} a_{mn} |m\ n\rangle$. Therefore, the unidirectional operator is satisfied at a node when

$$\hat{U}_+ \sum_{m,n} a_{mn} |m\ n\rangle = 0. \tag{107}$$

Since a state must be normalizable to have physical meaning, we next check that the states satisfying the unidirectional operator are normalizable. Recall from one-dimensional quantum mechanics that a wavefunction is normalized when

$$\langle\psi|\psi\rangle = \int_{-\infty}^{\infty} |\psi|^2\, dx = 1. \tag{108}$$

A wavefunction is normalized so that $|\psi|^2$ is a probability distribution and the probability of finding the particle somewhere is 1. The requirement for an unnormalized wavefunction $|\phi\rangle$ to be normalizable is that $\int_{-\infty}^{\infty} |\phi|^2\, dx$ is finite. If $|\phi\rangle$ is normalizable, then it can be multiplied be some constant to make equation (108) true. In loop quantum gravity, the requirement for a state to be normalizable is similar. We require

$$\sum_{m,n} |a_{mn}|^2 < \infty \tag{109}$$

meaning that $\sum_{m,n} |a_{mn}|^2$ converges. For each version of $\hat{U}_+$ that we create, we find solutions to equation (107) and check that these solutions are normalizable.

13. Setup for Quantization

In this section, we explain the process by which we create two quantum operators from the x-component of the connection K_x. The quantum operators created from K_y are similar, so we simply state them. We start by creating two operators $\hat{K}_x^{(1)}$ and $\hat{K}_x^{(2)}$ by using two different approximations for K_x. The superscript number in parentheses simply indicates which version of $\hat{K}_x$ it is. We then modify $\hat{K}_x^{(1)}$ to $\hat{\mathcal{K}}_x^{(1)}$ and $\hat{K}_x^{(2)}$ to $\hat{\mathcal{K}}_x^{(2)}$ by eliminating extra factors and ensuring real coefficients. To begin, we use a Taylor expansion of $h_v[K_x] = e^{i\frac{\mu_0}{2}K_x}$. We find

$$K_x \approx \frac{h_v[K_x] - h_v^{-1}[K_x]}{i\mu_0} = \frac{2}{\mu_0}\sin\left(\frac{\mu_0 K_x}{2}\right) \tag{110}$$

and

$$K_x \approx \frac{h_v[K_x] - 1}{i\frac{\mu_0}{2}}. \tag{111}$$

We use these approximations to define $\hat{K}_x$. Using approximation (110), we define

$$\hat{K}_x^{(1)} := \frac{-i}{\mu_0}\left(e^{i\frac{\mu_0}{2}K_x} - e^{-i\frac{\mu_0}{2}K_x}\right). \tag{112}$$

Acting on a state $|\mu\rangle = e^{i\frac{\mu}{2}K_x}$, we find

$$\begin{aligned}\hat{K}_x^{(1)}|\mu\rangle &= \frac{-i}{\mu_0}\left(e^{i\frac{\mu_0}{2}K_x} - e^{-i\frac{\mu_0}{2}K_x}\right)e^{i\frac{\mu}{2}K_x} \\ &= \frac{-i}{\mu_0}\left(e^{i\frac{\mu+\mu_0}{2}K_x} - e^{i\frac{\mu-\mu_0}{2}K_x}\right) \\ &= \frac{-i}{\mu_0}\left(|\mu+\mu_0\rangle - |\mu-\mu_0\rangle\right).\end{aligned} \tag{113}$$

Classically, K_x is real, so one would expect the eigenvalues of $\hat{K}_x$ to also be real. We renormalize $\hat{K}_x$ so its eigenvalues will be real as well as to remove any stray factors. Thus,

$$\hat{\mathcal{K}}_x^{(1)}|\mu\rangle = \frac{-\mu_0}{i}\hat{K}_x^{(1)}|\mu\rangle = |\mu+\mu_0\rangle - |\mu-\mu_0\rangle. \tag{114}$$

Doing the same process with approximation (111), we define

$$\hat{K}_x^{(2)} := \frac{-2i}{\mu_0}\left(e^{i\frac{\mu_0}{2}K_x} - 1\right) \tag{115}$$

to find

$$\begin{aligned}\hat{K}_x^{(2)}|\mu\rangle &= \frac{-2i}{\mu_0}\left(e^{i\frac{\mu_0}{2}K_x} - 1\right)e^{i\frac{\mu}{2}K_x} \\ &= \frac{-2i}{\mu_0}\left(e^{i\frac{\mu+\mu_0}{2}K_x} - e^{i\frac{\mu}{2}K_x}\right) \\ &= \frac{-2i}{\mu_0}\left(|\mu+\mu_0\rangle - |\mu\rangle\right).\end{aligned} \tag{116}$$

Renormalization gives

$$\hat{\mathcal{K}}_x^{(2)}|\mu\rangle = \frac{-\mu_0}{2i}\hat{K}_x^{(2)}|\mu\rangle = |\mu+\mu_0\rangle - |\mu\rangle. \tag{117}$$

We will refer to equation (114) as version 1 of $\hat{\mathcal{K}}_x$ and to equation (117) as version 2 of $\hat{\mathcal{K}}_x$. Versions 1 and 2 of $\hat{\mathcal{K}}_y$ are similar:

$$\hat{\mathcal{K}}_y^{(1)} |\nu\rangle = |\nu + \nu_0\rangle - |\nu - \nu_0\rangle \tag{118}$$

$$\hat{\mathcal{K}}_y^{(2)} |\nu\rangle = |\nu + \nu_0\rangle - |\nu\rangle . \tag{119}$$

We now have two versions of $\hat{\mathcal{K}}_x$ and $\hat{\mathcal{K}}_y$ to use to quantize the unidirectional constraint. We will quantize each term of the constraint separately. For the first term, we have two choices. The first choice is

$$\begin{aligned}\int_I dz f K_x E^x &\longrightarrow \widehat{\int_I dz f \mathcal{K}_x E^x} \\ &= \sum_{v\in I} f(v) \hat{\mathcal{K}}_x \hat{E}^x .\end{aligned} \tag{120}$$

The second choice, which is less conventional, is

$$\begin{aligned}\int_I dz f E^x K_x &\longrightarrow \widehat{\int_I dz f E^x \mathcal{K}_x} \\ &= \sum_{v\in I} f(v) \hat{E}^x \hat{\mathcal{K}}_x .\end{aligned} \tag{121}$$

We will refer to equation (120) as "E to the right" and to equation (121) as "E to the left." The second term has two similar versions using $\hat{\mathcal{K}}_y$ in place of $\hat{\mathcal{K}}_x$. The quantization of the third term is simple by comparison. For $\hat{\mathcal{E}}'$, we have action on nodes:

$$\begin{aligned}\int_I dz f \mathcal{E}' &\longrightarrow \widehat{\int_I dz f \mathcal{E}'} \\ &= f\hat{\mathcal{E}}(I_-) - f\hat{\mathcal{E}}(I_+) \\ &= f\frac{\gamma \ell_p^2}{4}(k_- - k_+).\end{aligned} \tag{122}$$

Equation (122) is used for the quantization of the third term in all cases. Now, we will go through the different cases we investigated without Pauli Matrices.

14. E to the Left, Version 2 of $\hat{\mathcal{K}}_x$

To find the first term of $\hat{U}_+$, we will act on a state $|\mu\rangle = e^{i\frac{\mu}{2}K_x}$ with the operator from equation (121) using equation (117) for $\hat{\mathcal{K}}_x$. Recall that the eigenvalues of $\hat{E}^x$ are given in equation (98). Then, acting with $\hat{\mathcal{K}}_x$ before $\hat{E}^x$ gives

$$\begin{aligned}\sum_{v\in I} f(v) \hat{E}^x \hat{\mathcal{K}}_x^{(2)} |\mu\rangle &= \sum_{v\in I} f(v) \hat{E}^x \left(|\mu + \mu_0\rangle - |\mu\rangle\right) \\ &= \frac{\gamma \ell_p^2}{2} \sum_{v\in I} f(v) \left((\mu + \mu_0) |\mu + \mu_0\rangle - \mu |\mu\rangle\right).\end{aligned} \tag{123}$$

Another way to express this operator is by its action on a state $|m\rangle = e^{i\frac{m}{2}K_x}$ where $m = \frac{\mu}{\mu_0}$ and $h_v[K_x] = e^{i\frac{1}{2}K_x}$. Then,

$$\sum_{v\in I} f(v) \hat{E}^x \hat{\mathcal{K}}_x^{(2)} |m\rangle = \frac{\gamma \ell_p^2}{2} \sum_{v\in I} f(v) \left((m+1) |m+1\rangle - m |m\rangle\right). \tag{124}$$

The quantization for the second terms proceeds similarly. We find

$$\sum_{v\in I} f(v)\hat{E}^y\hat{\mathcal{K}}_y^{(2)}\,|n\rangle = \frac{\gamma\ell_p^2}{2}\sum_{v\in I} f(v)\left((n+1)\,|n+1\rangle - n\,|n\rangle\right). \tag{125}$$

From now on, we denote states at a vertex by $|m\ n\rangle$ with the understanding that we mean $|m\ n\ k_+\ k_-\rangle$. Thus, our quantized version of $\int_I dz\, U_+[f]$ an operator where the result of its action is

$$\frac{\gamma\ell_p^2}{2}\sum_{v\in I} f(v)\left((m+1)\,|m+1\ n\rangle - m\,|m\ n\rangle + (n+1)\,|m\ n+1\rangle - n\,|m\ n\rangle\right) - f\frac{\gamma\ell_p^2}{4}(k_- - k_+)\,|m\ n\rangle. \tag{126}$$

We will now present states which satisfy this operator.

14.1. **Trivial Solution.** Consider the case where

$$k_- = k_+ \tag{127}$$

so that k is constant. Furthermore, to satisfy equation (107) at each node,

$$\frac{\gamma\ell_p^2}{2}\sum_{m,n} a_{mn}\left((m+1)\,|m+1\ n\rangle - m\,|m\ n\rangle + (n+1)\,|m\ n+1\rangle - n\,|m\ n\rangle\right) = 0 \tag{128}$$

$$\sum_{m,n}\left(a_{(m-1)n}m\,|m\ n\rangle - a_{mn}m\,|m\ n\rangle + a_{m(n-1)}n\,|m\ n\rangle - a_{mn}n\,|m\ n\rangle\right) = 0 \tag{129}$$

$$ma_{(m-1)n} - ma_{mn} + na_{m(n-1)} - na_{mn} = 0. \tag{130}$$

For amplitudes corresponding to $m=0$, equation (130) reduces to $a_{0n} = a_{0(n-1)}$ which implies that these amplitudes are constant. Similarly, the amplitudes are constant for $n=0$. Therefore, every $m=0$ amplitude and every $n=0$ amplitude must be zero in order for $\sum_{m,n}|a_{mn}|^2$ to converge.

For $a_{01} = a_{10} = 0$, it follows from equation (130) that $a_{11} = 0$. It then follows that $a_{21} = a_{12} = 0$. In fact, $a_{mn} = 0$ for all m, n. Thus, for $k_- = k_+$, the only solution is the trivial solution. This solution corresponds to degenerate space.

14.2. **Nontrivial Solution.** If instead, $k_- \neq k_+$, then

$$\sum_{m,n} a_{mn}\left((m+1)\,|m+1\ n\rangle - m\,|m\ n\rangle + (n+1)\,|m\ n+1\rangle - n\,|m\ n\rangle - \frac{1}{2}(k_- - k_+)\,|m\ n\rangle\right) = 0 \tag{131}$$

$$\sum_{m,n}\left(a_{(m-1)n}m\,|m\ n\rangle - a_{mn}m\,|m\ n\rangle + a_{m(n-1)}n\,|m\ n\rangle - a_{mn}n\,|m\ n\rangle - \frac{a_{mn}}{2}(k_- - k_+)\,|m\ n\rangle\right) = 0 \tag{132}$$

$$ma_{(m-1)n} - ma_{mn} + na_{m(n-1)} - na_{mn} - \frac{a_{mn}}{2}(k_- - k_+) = 0. \tag{133}$$

Assume $k_- - k_+$ is constant and let $\Delta k = k_- - k_+$. For amplitudes with $m = 0$ and $n \in \mathbb{N}$, equation (133) becomes

$$a_{0n} = \frac{n}{n + \Delta k/2}a_{0(n-1)}. \tag{134}$$

Written as an explicit formula, equation (134) is

$$a_{0n} = \frac{n!}{\prod_{i=1}^{n}(i + \Delta k/2)}a_{00} = \frac{\Gamma(1+\Delta k/2)\Gamma(n+1)}{\Gamma(n+1+\Delta k/2)}a_{00}. \tag{135}$$

Assuming Δk is an even natural number ($\Delta k \in 2\mathbb{N}$),

$$a_{0n} = \frac{(\Delta k/2)!\ n!}{(n+\Delta k/2)!}a_{00}. \tag{136}$$

Observe that

$$\sum_{n=0}^{\infty} |a_{0n}|^2 = \sum_{n=0}^{\infty} \left| \frac{(\Delta k/2)!\ n!}{(n+\Delta k/2)!} a_{00} \right|^2 = (\Delta k/2!\ a_{00})^2 \sum_{n=0}^{\infty} \left(\frac{n!}{(n+\Delta k/2)!} \right)^2. \tag{137}$$

Note that $\frac{n!}{(n+\Delta k/2)!} < \frac{1}{n}$ so $\left(\frac{n!}{(n+\Delta k/2)!}\right)^2 < \frac{1}{n^2}$. By the comparison test, equation (137) converges for all $\Delta k \in 2\mathbb{N}$.

After a bit more work, we find an explicit formula for a_{mn} that satisfies equation (133) for $m, n \in \mathbb{N}_0$. The amplitudes are given by

$$a_{mn} = \frac{(\Delta k/2)!(m+n)!}{(\Delta k/2+m+n)!} a_{00}. \tag{138}$$

We claim that $\sum_{n,m} |a_{mn}|^2$ converges. Observe:

$$\begin{aligned}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} |a_{mn}|^2 &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \left| \frac{(\Delta k/2)!(m+n)!}{(\Delta k/2+m+n)!} a_{00} \right|^2 \\ &= ((\Delta k/2)!\ |a_{00}|)^2 \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \left(\frac{(m+n)!}{(m+n+\Delta k/2)!} \right)\end{aligned} \tag{139}$$

Notice that terms for the same value of $m+n$ are the same. For a given value of $m+n$, there are $m+n+1$ terms. Let $\ell = m+n$, then

$$\begin{aligned}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} |a_{mn}|^2 &= ((\Delta k/2)!\ |a_{00}|)^2 \sum_{\ell=0}^{\infty} (\ell+1) \left(\frac{\ell!}{(\ell+\Delta k/2)!} \right)^2 \\ &= ((\Delta k/2)!\ |a_{00}|)^2 \sum_{\ell=0}^{\infty} \frac{1}{(\ell+1)\prod_{j=0}^{\Delta k/2-2} (\ell+\Delta k/2-j)^2}.\end{aligned} \tag{140}$$

Note that the denominator, when written as a sum, contains $\ell^{\Delta k+1}$. So,

$$\frac{1}{(\ell+1)\prod_{j=0}^{\Delta k/2-2} (\ell+\Delta k/2-j)^2} < \frac{1}{\ell^{\Delta k+1}}. \tag{141}$$

Because $\sum_{\ell=0}^{\infty} \frac{1}{\ell^{\Delta k+1}}$ converges for $\Delta k \geq 1$, it follows that $\sum_{m,n=0}^{\infty} |a_{mn}|^2$ converges for $\Delta k \geq 1$.

Unfortunately, the gamma function is undefined for the negative integers, meaning that Δk cannot be a negative even integer because $\Gamma(\Delta k/2+1)$ must be defined. Therefore, at this point, we can conclude that our solution is valid for $m, n \geq 0$ for $\Delta k \in 2\mathbb{N}$.

So far we've only shown convergence for $m, n \geq 0$, and because the gamma function is undefined for the negative integers, solutions to equation (133) do not exist when $m+n < 0$. Next the step, which we have not done, is to consider solutions where $m+n \geq 0$, but $m < 0$ or $n < 0$.

Therefore, we can only say with certainty that our solution is valid for $m, n \geq 0$ with $\Delta k \in 2\mathbb{N}$. We could possibly avoid this problem by using $|m|$ and $|n|$ in equation (138).

15. E TO THE RIGHT, VERSION 2 OF $\hat{\mathcal{K}}_x$

Proceeding similarly to Section 14, we find

$$\begin{aligned}\sum_{v\in I} f(v)\hat{\mathcal{K}}_x^{(2)}\hat{E}^x|\mu\rangle &= \frac{\gamma\ell_p^2}{2}\sum_{v\in I} f(v)\mu\hat{\mathcal{K}}_x^{(2)}|\mu\rangle \\ &= \frac{\gamma\ell_p^2}{2}\sum_{v\in I} f(v)\mu\left(|\mu+\mu_0\rangle - |\mu\rangle\right).\end{aligned} \tag{142}$$

This result can also be expressed as

$$\sum_{v\in I} f(v)\hat{\mathcal{K}}_x^{(2)}\hat{E}^x|m\rangle = \frac{\gamma\ell_p^2}{2}\sum_{v\in I} f(v)m\left(|m+1\rangle - |m\rangle\right). \tag{143}$$

Therefore, the complete action of the constraint operator is given by

$$\frac{\gamma\ell_p^2}{2}\sum_{v\in I} f(v)\left(m\,|m+1\ n\rangle - m\,|m\ n\rangle + n\,|m\ n+1\rangle - n\,|m\ n\rangle\right) - f\frac{\gamma\ell_p^2}{4}(k_- - k_+)\,|m\ n\rangle. \tag{144}$$

To satisfy equation (107) at a vertex, we must have

$$\sum_{m,n} a_{mn}\left((m\,|m+1\ n\rangle - m\,|m\ n\rangle + n\,|m\ n+1\rangle - n\,|m\ n\rangle) - \frac{1}{2}(k_- - k_+)\,|m\ n\rangle\right) = 0. \tag{145}$$

Upon reindexing the sum, we find

$$\begin{aligned}\sum_{m,n}\Big((m-1)a_{(m-1)n}\,|m\ n\rangle - ma_{mn}\,|m\ n\rangle + (n-1)a_{m(n-1)}\,|m\ n\rangle \\ - na_{mn}\,|m\ n\rangle - \frac{1}{2}(k_- - k_+)a_{mn}\,|m\ n\rangle\Big) = 0\end{aligned} \tag{146}$$

so that each term must satisfy

$$(m-1)a_{(m-1)n} - ma_{mn} + (n-1)a_{m(n-1)} - na_{mn} - \frac{1}{2}(k_- - k_+)a_{mn} = 0. \tag{147}$$

From here, we find the recurrence relation

$$a_{mn} = \frac{(m-1)a_{(m-1)n} + (n-1)a_{m(n-1)}}{m+n+\Delta k/2}. \tag{148}$$

where $\Delta k = k_- - k_+$. There are a few things to note from this formula. First, $a_{11} = 0$, and therefore, $a_{m1} = 0$ for $m \geq 1$. This can be shown using induction and the fact that $\frac{(m-1)a_{(m-1)1}}{1+\Delta k/2}$. Similarly, $a_{1n} = 0$ for $n \geq 1$. It follows that $a_{mn} = 0$ when $m \geq 1$ and $n \geq 1$.

Also, notice that a_{mn} does not exist when $m + n + \Delta k/2 = 0$. Since m and n take one all integer values, we must have $\Delta k/2 \notin \mathbb{Z}$, so Δk must be odd.

15.1. **Trivial Solution.**

The trivial solution is simply $a_{mn} = 0$ for all $m, n \in \mathbb{Z}$. As noted above, Δk must be odd, but there are no other restriction on Δk for the trivial solution to hold.

15.2. **Nontrivial Solution.** One solution to equation (148) requires $a_{mn} = 0$ if $m \neq -1$ or $n \geq 0$. For $m = -1$ and $n < 0$, we have

$$a_{(-1)(-|n|)} = \frac{1}{|n|!} \prod_{i=3}^{|n|} (i - \Delta k/2)\, a_{(-1)(-1)}. \tag{149}$$

For convenience, let $b = -n$, so

$$a_{(-1)(-b)} = \frac{1}{b!} \prod_{i=3}^{b} (i - \Delta k/2)\, a_{(-1)(-1)} \tag{150}$$

$$= \frac{2}{b!} \frac{\Gamma(b+1-\Delta k/2)}{\Gamma(3-\Delta k/2)} a_{(-1)(-1)}. \tag{151}$$

We then need to check that the amplitudes are normalizable. Observe:

$$\sum_{m,n} |a_{mn}|^2 = |a_{(-1)(-1)}|^2 + \sum_{b=2}^{\infty} \left| \frac{2}{b!} \frac{\Gamma(b+1-\Delta k/2)}{\Gamma(3-\Delta k/2)} a_{(-1)(-1)} \right|^2 \tag{152}$$

$$= |a_{(-1)(-1)}|^2 + \left| \frac{2a_{(-1)(-1)}}{\Gamma(3-\Delta k/2)} \right|^2 \sum_{b=2}^{\infty} \left| \frac{\Gamma(b+1-\Delta k/2)}{b!} \right|^2. \tag{153}$$

We only care about whether or not the sum in equation (153) converges. To start, let $\Delta k = 3$. Then the sum becomes

$$\sum_{b=2}^{\infty} \left| \frac{\Gamma(b-\frac{1}{2})}{b!} \right|^2 = \sum_{b=2}^{\infty} \left| \frac{\Gamma(b+\frac{1}{2})}{b!(b-\frac{1}{2})} \right|^2 \tag{154}$$

$$= \sum_{b=2}^{\infty} \left| \frac{(2b)!\sqrt{\pi}}{4^b (b!)^2 (b-\frac{1}{2})} \right|^2. \tag{155}$$

Making use of Stirling's approximation, we find

$$\frac{(2b)!}{(b!)^2} \approx \frac{(2b)^{2b} e^{-2b} \sqrt{2\pi(2b)}}{\left(b^b e^{-b} \sqrt{2\pi b}\right)^2} = \frac{4^b}{\sqrt{\pi b}}. \tag{156}$$

Therefore, our sum is approximately

$$\sum_{b=2}^{\infty} \left| \frac{1}{(b-\frac{1}{2})\sqrt{b}} \right|^2 = \sum_{b=2}^{\infty} \frac{1}{b(b-\frac{1}{2})^2}. \tag{157}$$

which clearly converges because $\frac{1}{b(b-\frac{1}{2})^2} < \frac{1}{b^3}$, which converges.

For $\Delta k \geq 5$,

$$\Gamma(b+1-\Delta k/2) = \frac{\Gamma(b+\frac{1}{2})}{\prod_{i=1}^{(\Delta k-1)/2} (b-i+\frac{1}{2})}. \tag{158}$$

Therefore, the sum from equation (153) is

$$\sum_{b=2}^{\infty} \left| \frac{\Gamma(b+1-\Delta k/2)}{b!} \right|^2 = \sum_{b=2}^{\infty} \left| \frac{\Gamma\left(b+\frac{1}{2}\right)}{b! \prod_{i=1}^{(\Delta k-1)/2} (b-i+\frac{1}{2})} \right|^2 \tag{159}$$

$$\leq \sum_{b=2}^{\infty} \left| \frac{\Gamma(b+\frac{1}{2})}{b!(b-\frac{1}{2})} \right|^2. \tag{160}$$

Thus, by the comparison test, the sum converges, which shows that $\sum_{m,n} |a_{mn}|^2$ converges for $\Delta k \geq 5$.

Now we will show that $\sum_{m,n} |a_{mn}|^2$ diverges for $\Delta k \leq 1$. To start, let $\Delta k = 1$. The relevent part of equation (153) is once again the sum

$$\sum_{b=2}^{\infty} \left| \frac{\Gamma(b+1-\Delta k/2)}{b!} \right|^2 = \sum_{b=2}^{\infty} \left| \frac{\Gamma\left(b+\frac{1}{2}\right)}{b!} \right|^2 \tag{161}$$

$$= \sum_{b=2}^{\infty} \left| \frac{(2b)!\sqrt{\pi}}{4^b (b!)^2} \right|^2 . \tag{162}$$

Using Stirling's approximation gives

$$\sum_{b=2}^{\infty} \left| \frac{1}{\sqrt{b}} \right|^2 = \sum_{b=2}^{\infty} \frac{1}{b}, \tag{163}$$

which diverges. For $\Delta k \leq -1$, we have

$$\Gamma(b+1-\Delta k/2) = \Gamma\left(b+\frac{1}{2}\right) \prod_{i=0}^{-(\Delta k+1)/2} \Gamma(b-i-\Delta k/2) \tag{164}$$

$$\geq \Gamma\left(b+\frac{1}{2}\right). \tag{165}$$

Therefore,

$$\sum_{b=2}^{\infty} \left| \frac{\Gamma(b+1-\Delta k/2)}{b!} \right|^2 \geq \sum_{b=2}^{\infty} \left| \frac{\Gamma\left(b+\frac{1}{2}\right)}{b!} \right|^2 \tag{166}$$

so equation (153) diverges for $\Delta k \leq -1$.

In summary, we have shown that the amplitudes given by equation (149) satisfy this version of the quantized unidirectional constraint when Δk is odd. These amplitudes are normalizable only for $\Delta k \geq 3$.

16. E to the Left, Version 1 of $\widehat{\mathcal{K}}_x$

In this section and the next, we will use version 1 of $\widehat{\mathcal{K}}_x$ to quantize the unidirectional constraint, but we will not present full solutions. Observe:

$$\begin{aligned}\sum_{v\in I} f(v)\widehat{E}^x \widehat{\mathcal{K}}_x^{(1)} |\mu\rangle &= \sum_{v\in I} f(v)\widehat{E}^x \left(|\mu+\mu_0\rangle - |\mu-\mu_0\rangle\right) \\ &= \frac{\gamma\ell_p^2}{2} \sum_{v\in I} f(v) \left((\mu+\mu_0)\,|\mu+\mu_0\rangle - (\mu-\mu_0)\,|\mu-\mu_0\rangle\right)\end{aligned} \tag{167}$$

which can also be expressed as

$$\sum_{v\in I} f(v)\widehat{E}^x \widehat{\mathcal{K}}_x^{(1)} |m\rangle = \frac{\gamma\ell_p^2}{2} \sum_{v\in I} f(v) \left((m+1)\,|m+1\rangle - (m-1)\,|m-1\rangle\right). \tag{168}$$

So the quantized unidirectional constraint is

$$\frac{\gamma\ell_p^2}{2} \sum_{v\in I} f(v)((m+1)\,|m+1\ n\rangle - (m-1)\,|m-1\ n\rangle + (n+1)\,|m\ n+1\rangle - (n-1)\,|m\ n-1\rangle) - f\frac{\gamma\ell_p^2}{4}\,(k_- - k_+)\,|m\ n\rangle . \tag{169}$$

In order for the constraint to be satisfied at a node, we need

$$\sum_{m,n} a_{mn}\Bigg((m+1)\,|m+1\ n\rangle - (m-1)\,|m-1\ n\rangle + (n+1)\,|m\ n+1\rangle$$
$$- (n-1)\,|m\ n-1\rangle - \frac{1}{2}\,(k_- - k_+)\,|m\ n\rangle\Bigg) = 0. \quad (170)$$

Upon simplication, we find

$$\sum_{m,n}\left(ma_{(m-1)n} - ma_{(m+1)n} + na_{m(n-1)} - na_{m(n+1)} - \frac{1}{2}\,(k_- - k_+)\,a_{mn}\right) = 0 \quad (171)$$

$$ma_{(m-1)n} - ma_{(m+1)n} + na_{m(n-1)} - na_{m(n+1)} - \frac{1}{2}\,(k_- - k_+)\,a_{mn} = 0. \quad (172)$$

17. E to the Right, Version 1 of $\widehat{\mathcal{K}}_x$

Observe:

$$\begin{aligned}\sum_{v\in I} f(v)\widehat{\mathcal{K}}_x^{(1)}\widehat{E}^x\,|\mu\rangle &= \frac{\gamma\ell_p^2}{2}\sum_{v\in I} f(v)\widehat{\mathcal{K}}_x^{(1)}\mu\,|\mu\rangle \\ &= \frac{\gamma\ell_p^2}{2}\sum_{v\in I} f(v)\mu\left(|\mu+\mu_0\rangle - |\mu-\mu_0\rangle\right).\end{aligned} \quad (173)$$

Alternatively, we can express the action as

$$\sum_{v\in I} f(v)\widehat{\mathcal{K}}_x^{(1)}\widehat{E}^x\,|m\rangle = \frac{\gamma\ell_p^2}{2}\sum_{v\in I} f(v)m\left(|m+1\rangle - |m-1\rangle\right). \quad (174)$$

Therefore, the unidirectional constraint is

$$\frac{\gamma\ell_p^2}{2}\sum_{v\in I} f(v)\Bigg(m\,|m+1\ n\rangle - m\,|m-1\ n\rangle + n\,|m\ n+1\rangle - n\,|m\ n-1\rangle\Bigg) - f\frac{\gamma\ell_p^2}{4}(k_- - k_+)\,|m\ n\rangle. \quad (175)$$

In order to satisfy the constraint, the amplitudes at each node must satisfy

$$\sum_{m,n} a_{mn}\left(m\,|m+1\ n\rangle - m\,|m-1\ n\rangle + n\,|m\ n+1\rangle - n\,|m\ n-1\rangle - \frac{1}{2}(k_- - k_+)\,|m\ n\rangle\right) = 0 \quad (176)$$

$$(m-1)a_{(m-1)n} - (m+1)a_{(m+1)n} + (n-1)a_{m(n-1)} - (n+1)a_{m(n+1)} - \frac{1}{2}(k_- - k_+)a_{mn} = 0. \quad (177)$$

18. Discussion

We found four possible versions of the unidirectional operator created from the unidirectional constraint and found states satisfying two of them. Some of these states were trivial, meaning that they correspond to degenerate space and are not significant. The other states that we found are normalizable provided that certain conditions are met. Both nontrivial solutions require $\Delta k \geq 2$, but the solution to one version requires Δk to be odd and the solution the other version requires Δk to be even. The significance of this discrepancy is unclear, however, we hope that further research will uncover additional solutions allowing Δk to take odd or even values for both versions of the constraint. Another difference to note is that our solution to one version of the constraint only works when we limit m and n to positive values (the first quadrant of a coordinate plane) and our solution to the other version only works when we consider all values of m and n (all four quadrants of a coordinate plane). In the latter case, only the third quadrant contains nonzero amplitudes. Since the value given by the volume operator depends only on the absolute values of m and n, neither case is immediately cause for concern, but allowing m and n to take on both positive and negative values leaves open more possibilities in the theory. All of our solutions are for a single vertex. A full solution would include a state for all vertices.

We also showed that using Pauli matrices in the quantization of K_x gives one of the same results that we obtained without Pauli matrices. Since using Pauli matrices gives the same result as $\hat{K}_x^{(1)}$, we conclude that $\hat{\mathcal{K}}_x^{(1)}$ is more consistent with the Gowdy model in $3+1$ dimensions than $\hat{\mathcal{K}}_x^{(2)}$ is. The solutions we found satisfy both versions of the unidirectional operator that use $\hat{\mathcal{K}}_x^{(2)}$, so future work will focus on finding solutions to the two versions of the unidirectional operator that use $\hat{\mathcal{K}}_x^{(1)}$.

Part 4. Conclusion

In Part 1, we represented of the diffeomorphism, Hamiltonian, and unidirectional constraints on a space-time diagram, showing their algebra visually. This representation mainly serves to help us better understand the constraints.

In Part 3, we presented our main results, quantization of the unidirectional constraint four different ways. We found normalizable solutions to the two quantizations given in sections 14 and 15. One key difference between the solutions is that one only worked for $m \geq 0$ and $n \geq 0$ while the other was zero for all values of m and n except for $m = -1$ and $n < 0$. The other key difference is that they required different values of Δk. We hope that further investigation will result in a more general solution for both quantizations, eliminating some of these differences. We also hope to find solutions for the other two quantizations, given in sections 16 and 17.

Finding nontrivial normalizable solutions to a unidirectional operator is a huge step forward in the theory. In addition to finding more solutions, the next step is to investigate the compatibility of the unidirectional operator with a diffeomorphism operator.

Part 5. Appendix

19. Quantization Using Pauli Matrices

Using Pauli matrices in the quantization of $\hat{K}_x$ and $\hat{K}_y$ allows us to compare our one-dimensional simplification with the full three-dimensional theory. In this appendix, we use the matrices $\tau_i = -\frac{i}{2}\sigma_i$ where the non-subscript i refers to the imaginary unit. We use the properties $\mathrm{Tr}\left[\sigma_i\right] = 0$ and $\sigma_i^2 = \mathbb{1}$, which imply $\mathrm{Tr}\left[\tau_i\right] = 0$ and $\tau_i^2 = -\frac{1}{4}\mathbb{1}$.

The main difference in the quantization is that we now use $h_v[K_x] = e^{\tau_x \mu_0 K_x}$. The Taylor expansion is $e^{\tau_x \mu_0 K_x} \approx \mathbf{1} + \tau_x \mu_0 K_x$. Using this Taylor expansion, we find two approximations for K_x:

$$K_x \approx -\frac{1}{\mu_0}\mathrm{Tr}\left[\tau_x\left(e^{\tau_x\mu_0K_x} - e^{-\tau_x\mu_0K_x}\right)\right] \tag{178}$$

and

$$K_x \approx -\frac{2}{\mu_0}\mathrm{Tr}\left[\tau_x\left(e^{\tau_x\mu_0K_x} - \mathbf{1}\right)\right]. \tag{179}$$

We can use both of these approximations to create an operator $\hat{K}_x$, which we define as

$$\hat{K}_x^{(3)} := -\frac{1}{\mu_0}\mathrm{Tr}\left[\tau_x\left(e^{\tau_x\mu_0K_x} - e^{-\tau_x\mu_0K_x}\right)\right] \tag{180}$$

and

$$\hat{K}_x^{(4)} := -\frac{2}{\mu_0}\mathrm{Tr}\left[\tau_x\left(e^{\tau_x\mu_0K_x} - \mathbf{1}\right)\right]. \tag{181}$$

As before, we will need to renormalize each version of $\hat{K}_x$ for which we will need the identity

$$e^{\tau_x\mu_0K_x} = \cos\left(\frac{\mu_0}{2}K_x\right) + 2\tau_x\sin\left(\frac{\mu_0}{2}K_x\right). \tag{182}$$

From equation (180), we find

$$\begin{aligned}
\hat{K}_x^{(3)}\left|\mu\right\rangle &= -\frac{1}{\mu_0}\mathrm{Tr}\left[\tau_x\left(e^{\tau_x\mu_0K_x} - e^{-\tau_x\mu_0K_x}\right)\right]e^{i\frac{\mu}{2}K_x} \\
&= -\frac{1}{\mu_0}\mathrm{Tr}\left[\tau_x\left(\cos\left(\frac{\mu_0}{2}K_x\right) + 2\tau_x\sin\left(\frac{\mu_0}{2}K_x\right) - \cos\left(-\frac{\mu_0}{2}K_x\right) - 2\tau_x\sin\left(-\frac{\mu_0}{2}K_x\right)\right)\right]e^{i\frac{\mu}{2}K_x} \\
&= -\frac{1}{\mu_0}\mathrm{Tr}\left[4\tau_x^2\sin\left(\frac{\mu_0}{2}K_x\right)\right]e^{i\frac{\mu}{2}K_x} \\
&= \frac{1}{\mu_0}\mathrm{Tr}\left[\sin\left(\frac{\mu_0}{2}K_x\right)\right]e^{i\frac{\mu}{2}K_x} \\
&= \frac{2}{\mu_0}\sin\left(\frac{\mu_0}{2}K_x\right)e^{i\frac{\mu}{2}K_x} \\
&= \frac{2}{\mu_0}\frac{e^{i\frac{\mu_0}{2}K_x} - e^{-i\frac{\mu_0}{2}K_x}}{2i}e^{i\frac{\mu}{2}K_x} \\
&= \frac{1}{i\mu_0}\left(e^{i\frac{\mu+\mu_0}{2}K_x} - e^{i\frac{\mu-\mu_0}{2}K_x}\right) \\
&= \frac{1}{i\mu_0}\left(\left|\mu+\mu_0\right\rangle - \left|\mu-\mu_0\right\rangle\right).
\end{aligned} \tag{183}$$

The action of this operator turns out to be the same as the action of $\hat{K}_x^{(1)}$, so we will not bother renormalizing.

For $\hat{K}_x^{(4)}$, we find

$$
\begin{aligned}
\widehat{K}_x \left|\mu\right\rangle &= -\frac{2}{\mu_0}\mathrm{Tr}\left[\tau_x\left(e^{\tau_x\mu_0 K_x}-\mathbf{1}\right)\right]e^{i\frac{\mu}{2}K_x} \\
&= -\frac{2}{\mu_0}\mathrm{Tr}\left[\tau_x e^{\tau_x\mu_0 K_x}\right]e^{i\frac{\mu}{2}K_x} \\
&= -\frac{2}{\mu_0}\mathrm{Tr}\left[\tau_x\cos\left(\frac{\mu_0}{2}K_x\right)+2\tau_x^2\sin\left(\frac{\mu_0}{2}K_x\right)\right]e^{i\frac{\mu}{2}K_x} \\
&= -\frac{2}{\mu_0}\mathrm{Tr}\left[-\frac{1}{2}\sin\left(\frac{\mu_0}{2}K_x\right)\right]e^{i\frac{\mu}{2}K_x} \\
&= \frac{2}{\mu_0}\sin\left(\frac{\mu_0}{2}K_x\right)e^{i\frac{\mu}{2}K_x} \\
&= \frac{2}{\mu_0}\frac{e^{i\frac{\mu_0}{2}K_x}-e^{i\frac{\mu_0}{2}K_x}}{2i}e^{i\frac{\mu}{2}K_x} \\
&= \frac{1}{i\mu_0}\left(e^{i\frac{\mu+\mu_0}{2}K_x}-e^{i\frac{\mu-\mu_0}{2}K_x}\right) \\
&= \frac{1}{i\mu_0}\left(\left|\mu+\mu_0\right\rangle-\left|\mu-\mu_0\right\rangle\right).
\end{aligned}
\tag{184}
$$

Thus, we see that with the use of Pauli matrices, both versions $\hat{K}_x^{(3)}$ and $\hat{K}_x^{(4)}$ yield the same result as $\hat{K}_x^{(1)}$. Therefore, it suffices to use equation $\hat{K}_x^{(1)}$ and forgo the use of Pauli matrices because they do not provide any new information.

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